MA 312 Commutative Algebra / Aug–Dec 2017

(Int PhD. and Ph. D. Programmes)

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Lectures : Wednesday and Friday ; 14:00–15:30 **Venue:** MA LH-2 (**if LH-1 is not free**) / LH-1

Seminars : Sat, Nov 18 (10:30–12:45) ; Sat, Nov 25 (10:30-12:45)

Final Examination : Tuesday, December 05, 2017, 09:00–12:00

Evaluation Weightage : Assignments : 20% **Seminars :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)

	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	76 — 90	61 — 75	46 — 60	35 — 45	< 35

	Grade A$^+$	Grade A	Grade B$^+$	Grade B	Grade C	Grade D	Grade F
Marks-Range	> 90	81 — 90	71 — 80	61 — 70	51 — 60	40 — 50	< 40

3. Rings and Modules with Chain Conditions

Submit a solutions of $*$-Exercises ONLY. **Due Date : Wednesday, 13-09-2017**

3.1 Let A be a ring and V_i, $i \in I$, be a family of A-modules $V_i \neq 0$. If I not finite, then the A-module $\bigoplus_{i\in I} V_i$ is neither noetherian nor artinian.

3.2 Let I be an infinite set and A be a ring $\neq 0$. The product ring A^I is neither noetherian nor artinian.

3.3 Let k be a field.
(a) Let $B = k[x]$ be a cyclic k-algebra. Then every k-subalgebra A of B is a finite type k-algebra. (**Hint :** If $f \in A$, $f = \sum_{i=0}^{m} a_i x^i$, $a_m \neq 0$, $m \geq 1$, then $B = \sum_{i=0}^{m-1} k[f]x^i$ is a finite over $k[f] \subseteq A$.)

(b) Let $B = k[\mathbb{N}^2]$ be the monoid algebra over k of the additive monoid $\mathbb{N}^2$ and let $X := e_{(1,0)}$, $Y := e_{(0,1)}$. Then $B = k[X,Y]$, and the monomials $X^iY^j = e_{(i,j)}$, $(i,j) \in \mathbb{N}^2$, form a k-basis of B. Let A be the k-subalgebra of B generated by the monomials $X, X^2Y, \ldots, X^{n+1}Y^n, \ldots$. Then A is not a noetherian ring, much less than a finite type k-algebra. (**Hint :** Note that B is the polynomial algebra in two indeterinates X,Y over k and $X^{n+1}Y^n$ does not belong to the ideal (in A) generated by $X, \ldots, X^nY^{n-1}$, for every $n \in \mathbb{N}$.)

3.4 Let A be a ring in which every ideal has a generating system consisting of r elements. If V is an A-module generated by n elements, then every submodule U of V has a genarting system of cardinality nr.In particular, over a PID every submodule of a module with generating system of cardinality n is also generated by n elements. (**Hint :** By induction on n. Suppose $V = Ax_1 + \cdots + Ax_n$ and $f : V \to V/Ax_1$ is the residue-class map, then consider the restriction map $f|U : U \to V/Ax_1$. See[1])

3.5 Let V be amodule over the noetherian ring A with a generating system x_i, $i \in I$, where I is infinite. Then every submodule U of V has a generating system for the form y_i, $i \in I$.

3.6 Let A be a ring, V an A-module, $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ be a chain of submodule of V and f be an endomorphism of V with $f(V_i) \subseteq V_i$ for all $i = 1, \ldots, n$. Let $f_i : V_i/V_{i-1} \to V_i/V_{i-1}$, $i = 1, \ldots, n$, denote the endomorphism induced by f. Then

(a) If all but endomorphisms $f, f_1, \ldots, f_n$ are automorphisms, then all are automorphisms.

(b) If f is an automorphism, then all $f_1, \ldots, f_n$ are automorphisms if any one of the following condition is satisfied : (1) V/V_1 is noetherian. (2) $V_2/V_1, \ldots, V_n/V_{n-1}$ are finite A-modules. (3) V_{n-1} is artinian.

3.7 Let A be a noetherian ring. Then every surjective ring endomorphism of A is an automorphism.

3.8 Let A be a finite type (commutative) algebra over the ring R. Then every surjective R-algebra endomorphism φ of A is an automorphism. (**Hint :** Suppose that $\varphi(x) = 0$ and $x_1, \ldots, x_m$ ia a R-algebra generating system for A. The construct a finitely generated $\mathbb{Z}$-subalgebra R' of R such that $R'[x_1, \ldots, x_m]$ contain x as well as φ is a surjective endomorphism $R'[x_1, \ldots, x_m]$. — Note that the assertion does not hold for arbitrary commutative algebra. Examples!)

3.9 Let V be a module over a ring A and U be a submodule $\neq 0$ of V. If V noetherian or if V finite, then V and V/U are not isomorphic as A-modules. (**Hint :** If they are isomorphic the give a surjective surjective A-endomorphism of V with kernel U.)

3.10 Let $\mathfrak{a}$ be a non-zero ideal is a noetherian ring A. Then A and $A/\mathfrak{a}$ are not isomorphic rings.

3.11 Let $\mathfrak{a}$ be a non-zero ideal in a finite type commutative algebra A over the ring R. Then A and $A/\mathfrak{a}$ are not isomorphic as R-algebras.

3.12 Let K be a field, $I := \mathbb{N} \cup \{\infty\}$, $V := K^{(I)}$ and e_i, $i \in I$, be the standard basis of V and $V_n := \sum_{i=0}^{n} Ke_i$ for $n \in \mathbb{N}$, $V_\infty := \sum_{i\in\mathbb{N}} Ke_i$. The set of K-endomorphisms f of V with $f(V_n) \subseteq V_n$ for all $n \in \mathbb{N}$ is a K-subalgebra

[1] Note that if V_1, V_2 and U are submodules of V with $V_1 \subseteq V_2$. Then $(V_2 \cap U)/(V_1 \cap U)$ is isomorphic to a submodule of V_2/V_1, and $(V_2 + U)/(V_1 + U)$ is isomorphic to a residue-class module of V_2/V_1.

A of $\operatorname{End}_K V$. With respect to the natural A-module structure on V, besides 0 and V, V_n, $n \in \mathbb{N}$, and V_∞ are the only A-submodules of V. The A-module $V(=Ae_\infty)$ is cyclic and artinian, but not noetherian.

3.13 Let A be a commutative ring.

(a) Let V be a finite A-module and W a noetherian (resp. artinian) A-module. Then $\operatorname{Hom}_A(V,W)$ is also noetherian (resp. artinian).

(b) Let V be an A-module which is noetherian (resp. finite and artinian). Then $\operatorname{End}_A V$ is a noetherian (resp. finite and artinian) A-module. In particular, every A-subalgebra of $\operatorname{End}_A V$ is noetherian (resp. finite artinian).

3.14 Let A be a noetherian ring and B be an A-algebra of finite type. Let $\mathfrak{b}$ be an ideal in B such that the residue-class algebra $B/\mathfrak{b}$ is finite over A. Then $\mathfrak{b}$ is a finitely generated ideal in B, and for every $n \in \mathbb{N}$, the residue-class algebra $B/\mathfrak{b}^n$ is finite over A.
(**Hint:** There exists an A-algebra generating system $b_1,\ldots,b_m$ of B with $B = \mathfrak{b} + Ab_1 + \cdots + Ab_m$ and there exist elements $a_{ij}^k \in A$ with $a_{ij} := b_ib_j - \sum_{k=1}^m a_{ij}^k b_k \in \mathfrak{b}$ for $1 \le i, j \le m$. FoÂĄr the ideal $\mathfrak{c}(\subseteq \mathfrak{b})$ generated by the a_{ij}, $1 \le i, j \le m$, it follows that $B = \mathfrak{c} + Ab_1 + \cdots + Ab_m$. Further, it follows that $\mathfrak{b}$ is finitely generated and hence the $\mathfrak{b}^n/\mathfrak{b}^{n+1}$, $n \in \mathbb{N}$, are finite A-modules.)