MA 312 Commutative Algebra / Aug–Dec 2017

(Int PhD. and Ph. D. Programmes)

Download from : `http://www.math.iisc.ernet.in/patil/courses/courses/Current Courses/...`

Tel : +91-(0)80-2293 3212/09449076304 **E-mails :** patil@math.iisc.ernet.in

Lectures : Wednesday and Friday ; 14:00–15:30 **Venue:** MA LH-2 (**if LH-1 is not free**) / LH-1

Seminars : Sat, Nov 18 (10:30–12:45) ; Sat, Nov 25 (10:30-12:45)

Final Examination : Tuesday, December 05, 2017, 09:00–12:00

Evaluation Weightage : Assignments : 20% **Seminars :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)

	Grade S	Grade A	Grade B	Grade C	Grade D	Grade F
Marks-Range	>90	$76-90$	$61-75$	$46-60$	$35-45$	<35

	Grade A^+	Grade A	Grade B^+	Grade B	Grade C	Grade D	Grade F
Marks-Range	>90	$81-90$	$71-80$	$61-70$	$51-60$	$40-50$	<40

6. Rings and Modules of Fractions — Localisation

6.1 Let A be a commutative ring, $S \subseteq A$ be a multiplicative system in A and let V be an A-module. We say that an element $a \in A$ is a non-zero divisor on V, if the map $\lambda_a : V \to V$, $x \mapsto ax$ is injective. If $a \in A$ is a non-zero divisor on V, then $a/1$ is a non-zero divisor on the A_S-module V_S. In particular, if $a \in A$ is a non-zero divisor in A, then $a/1$ is a non-zero divisor in A_S.

6.2 Let A be a commutative ring and let $T \subseteq S \subseteq A$ be multiplicative systems in A. Then the A_T-algebra A_S is canonically isomorphic to the ring of fractions of A_T with respect to the image of S in A_T under the canonical map $\iota_T : A \to A_T$, i.e. $A_S \cong (A_T)_{\iota_T(S)}$ as A_T-algebras.

6.3 (so(Saturated multiplicative systems) Let A be a commutative ring. For every multiplicative system $S \subseteq A$, the subset $S' := \{a \in A \mid a \ \textit{divide some} \ \ s \in S\}$ is a multiplicative system in A and $S' = \iota_S^{-1}((A_S)^\times)$, where $\iota_S : A \to A_S$ is the canonical ring homomorphism $a \mapsto a/1$. Further, $S \subseteq S'$ and $(S')' = S'$. If S and T are multiplicative systems in A, then A_S and A_T are isomorphic A- algebras if and only if $S' = T'$.
(**Remark :** The multiplicative system S' is called the satuartion of S ; S is called saturated if $S = S'$.)

6.4 Let A be a non-zero commutative ring, S be the multiplicative system of non-zero divisors in A. Then the total quotient ring $\mathrm{Q}(A) := A_S$ is a non-zero ring. An A-module V is called module with rank over A if V_S is a free A_S-module; in this case, we also say that V has rank over A and put :

$$\mathrm{Rank}_A V := \mathrm{Rank}_{A_S} V_S \,.$$

(a) Every free A-module V is a module with rank over A and in this case rank is nothing but the rank of the free A-module V, i.e. *the* cardinality of an A-basis of V.

(b) If the A-module V is a module with rank over A, then the A_S-module V_S is a module with rank over A_S. In fact, V_S has an A_S-basis of the form $x_i/1\,, i \in I$; moreover, $x_i\,, i \in I$, is a maximal linearly independent family in V.

(c) Suppose that A is an integral domain. Then $S = A \setminus \{0\}$ and $\mathrm{Q}(A) = A_S$ is the quotient field of A and hence every A_S-module is free. In particular, every A-module V has rank over A; moreover, $\mathrm{Rank}_A V$ is *the* cardinality of a maximal linearly independent family in V.

6.5 Let A be a commutative ring, S be the multiplicative system of non-zero divisors in A and let V be an A-module. Then the kernel of the canonical map $\iota_S^V : V \to V_S$ is the torsion submodule

$$\mathrm{t}_A V := \{x \in V \mid sx = 0 \ \text{ for some } \ s \in S\}$$

of V. We say that V is torsion (respectively torsion-free) if $\mathrm{t}_A V = V$ (respectively $\mathrm{t}_A V = 0$). Show that :

(a) If V is finitely generated torsion-free with rank over A, then V is isomorphic to a submodule of a finite free A-module

(b) For an abelain group H, the following are equivalent :

(i) H is isomorphic to a subgroup of the additive group $(\mathbb{R}, +)$.

(ii) H is torsion free and $\mathrm{Card}(H) \leq \mathrm{Card}(\mathbb{R})$. (**Hint :** If $\mathrm{Card}(H) \leq \mathrm{Card}(\mathbb{R})$ then $\mathrm{Rank}(H) \leq \mathrm{Card}(\mathbb{R})$.)

6.6 Let A be a commutative ring, S a multiplicative system in A and V, W be modules over A. Then we have the canonical homomorphism

$$\Phi : \mathrm{Hom}_A(V, W)_S \to \mathrm{Hom}_{A_S}(V_S, W_S)\,,$$

Show that :

(a) If V is a finite A-module, then Φ injective.

(b) If V is a finite A-module and if the canonical homomorphism $W \to W_S$ injective (in this case one say that W is S-torsion-free), then Φ bijective.

(c) If V is finitely presented A-module, then Φ bijective.

(**Hint :** Consider an exact sequence $G \xrightarrow{f} F \xrightarrow{g} V \longrightarrow 0$ with finite free A-modules F, G and the canonical commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_A(V,W)_S & \xrightarrow{g'} & \mathrm{Hom}_A(F,W)_S & \xrightarrow{g} & \mathrm{Hom}_A(G,W)_S \\ & \Phi\downarrow & & \Phi_F\downarrow & & \Phi_G\downarrow \\ 0 \longrightarrow & \mathrm{Hom}_{A_S}(V_S,W_S) & \xrightarrow{f'} & \mathrm{Hom}_{A_S}(F_S,W_S) & \xrightarrow{f} & \mathrm{Hom}_{A_S}(G_S,W_S) \end{array}$$

with exact rows, Φ_F, Φ_G are bijective and hence Φ is bijective.)

6.7 **(a)** Let A be an integral domain, $S := A \setminus \{0\}$ and $K := A_S = \mathrm{Q}(A)$. Show that if the canonical homomorphism

$$\mathrm{Hom}_A(K,A)_S \longrightarrow \mathrm{Hom}_{A_S}(K_S,A_S)$$

is surjective, then $A = K$.(**Hint :** Consider id_K!.) - Deduce that if K is finite over A then $A = K$.

(b) Let A be a commutative ring and S be a multiplicative system in A. If A_S is a finite A-module, then A_S is isomorphic to $A/\mathrm{Ker}\,(A \to A_S)$.

6.8 Let K be a field, I be an infinite indexed set and $A := K^I$. Further, let $\mathfrak{a}$ be the ideal $K^{(I)}$ in A and S be the multiplicative system of elements $(s_i) \in K^I$ with $s_i \neq 0$ for almost all $i \in I$. Then :

(a) The canonical homomorphism

$$\mathrm{Hom}_A(A/\mathfrak{a},A)_S \longrightarrow \mathrm{Hom}_{A_S}((A/\mathfrak{a})_S,A_S)$$

is not surjective. (**Hint :** In fact, $\mathrm{Ann}_A \mathfrak{a} = 0$, $\mathrm{Hom}_A(A/\mathfrak{a},A) \cong \mathrm{Ann}_A \mathfrak{a}$, $f \mapsto f(1_{A/\mathfrak{a}})$ and $\mathfrak{a}_S = 0$.)

(b) For every infinite set J the canonical homomorphism

$$\mathrm{Hom}_A(A^{(J)},A)_S \longrightarrow \mathrm{Hom}_{A_S}(A_S^{(J)},A_S)$$

is not injective.

6.9 Let A be a commutative ring and let V be a projective A- module (i. e. V is a direct summand of a free A-module). Let S be the multiplicative system of non-zero divisors in A. If V_S is a finite A_S-module, then V is a finite A-module. (**Hint :** Let f be an embedding of V as a direct summand in a free A- module of the type $A^{(I)}$ and consider the image of f_S.) In particular, a projective module over an integral domain is finite if and only if it has a finite rank.

6.10 Let A be a commutative ring and $\mathfrak{a} \subseteq A$ be an ideal; S denote the multiplicative system $1+\mathfrak{a}$ in A. Then $\mathfrak{a}A_S$ is contained in the Jacobson-radical of A_S. (**Hint :** $1+\mathfrak{a}A_S \subseteq (A_S)^\times$.)

6.11 Let A be a commutative ring and V be a finite A-module. For a multiplicative system S in A, show that $V_S = 0$ if and only if $sV = 0$ for some $s \in S$.

6.12 (Lemma of Dedekind) Let A be a commutative ring, V be a finite A-module and $\mathfrak{a}$ be an ideal in A with $V = \mathfrak{a}V$. Show that $(1+a)V = 0$ for some $a \in \mathfrak{a}$. (**Hint :** Note that $V_{1+\mathfrak{a}} = 0$ by the Lemma of Krull-Nakayama[1] — **Remark :** For an another elementary proof : Suppose that $V = Ax_1 + \cdots + Ax_n$ and $V_i := Ax_1 + \cdots + Ax_i$ for $i = 0,\ldots,n$. By induction show that there are elements $a_j \in \mathfrak{a}$ such that $(1-a_j)V \subseteq \mathfrak{a}V_{n-j}$, $j = 0,\ldots,n$.)

[1] **Lemma of Krull-Nakayama** *Let A be a commutative ring, $\mathfrak{a}$ be an ideal in A. The following statements are equivalent* :
(i) $\mathfrak{a} \subseteq \mathfrak{m}_A$. (ii) *For every A-module V and every submodule U of V with V/U finitely generated, the following implication hold* : *If $V = U + \mathfrak{a}V$, then $V = U$.*