Lecture 1-1

Class No. 1

$\mathbb{N}$ = Set of natural numbers
$= \{0, 1, 2 \ldots \ldots\}$

$\mathbb{N}^* = \mathbb{N}_+ = \{1, 2, 3 \ldots \ldots\} = \{n \in \mathbb{N} \mid n \neq 0\}$
$= \mathbb{N} \setminus \{0\}$

Peano's Axioms

(1) A non-empty set $\mathbb{N}$ such that with a designated element $0 \in \mathbb{N}$.

(2) A successor map s with the following properties

(a) $0 \neq s(n) \;\forall\, n \in \mathbb{N}$. i.e, $0 \notin$ Image of s.

(b) S is injective i.e, $a = b \Leftrightarrow f(a) = f(b)$

(c) Axiom of induction:

If $X \subseteq \mathbb{N}$ (nonempty), $0 \in X$, then $a \in X \Rightarrow s(a) \in X$,

then $X = \mathbb{N}$.

Axiom of Induction allows to define addition and multiplication

$$+ : (m, n) \longrightarrow m + n$$
$$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \quad \text{(Identity 0)}$$

$$\cdot : (m, n) \longrightarrow mn$$
$$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \quad \text{(Identity 1)}$$

The operations are associative, commutative

$+$, and $\cdot$ are connected by distributive law

Monoid: A monoid $(A, +)$ is a pair of non-empty set A, and a binary operation $+$

$+ : A \times A \longrightarrow A$, such that the operation is associative with a identity element.

$(0, a) \longmapsto 0 + a = a$
$(a, 0) \longmapsto a + 0 = a$

Examples: $(\mathbb{N}, +)$, $(\mathbb{N}^*, \cdot)$

Invertible element: For $+$, a invertible is $-a$
$\cdot$, a invertible is a^{-1}

$A^{\times}$ = the set of all elements in A which have inverses

A Group is a monoid $A = A^{\times}$

Examples: $(\mathbb{Z}, +)$ is a group
$(\mathbb{Z}^*, \cdot)$ is not a group because
$(\mathbb{Z}^*)^{\times} = \{\pm 1\}$

$(\mathbb{Q}, +)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ ~~is~~ are groups.

More examples:
$X^{\times} = S(X) = \{ f : X \to X \mid f \text{ is bijective} \}$

X^X is the set of all maps (bijective) from X to X.

S(X) is also called the permutation group on X.

Note: $X \xrightarrow{f} X$, $X \xrightarrow{g} X$, $g \circ f : X \to X$.

We can show that $|S(X)| = |X|!$

↳ Cardinality of X

$\boxed{\text{Lecture 2-1}}$ Class No. 2

Examples of Monoids:

~~$(\mathbb{N}, +, \times)$~~, $(\mathbb{N}, +)$, $(\mathbb{N}^*, \cdot)$, $(\mathbb{Z}, +)$, $(\mathbb{Z}^*, \cdot)$

$(\mathbb{Q}, +)$, $(\mathbb{Q}^*, \cdot)$ $(\mathbb{R}, +)$, $(\mathbb{R}^*, \cdot)$ . . .

If X is the set, the power set of X is defined as

$$P(X) = \{ A \mid A \subseteq X \}.$$

The power set P(X) is a monoid with the operation union. The only inverse is ϕ.

$M^X = \{ x \in M \mid x \text{ is invertible, ie. } y \in M, \text{ s.t } x \cdot y = y \cdot x = e \}$

M^X is a group.

Another example: $(P(X), \cap)$

~~non-empty~~

X is the identity element in monoid.

$X^X = \text{Maps}(X, X)$ or $X^X = \{ f : X \to X \text{ maps} \}$

$\circ$ = Composition is a binary operation

Identity map is the ~~identy~~ identity element