Number-Theoretic Functions

Let $n \in \mathbb{N}^*$. The no: of positive divisors of n is denoted by $\tau(n)$.

$$\tau(n) = \prod_{i=1}^{r} (\alpha_i + 1) \quad \text{where } n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

$$p_1, p_2 \ldots p_r \in \mathbb{P}.$$

In particular, $\tau(p^\alpha) = \alpha + 1$

Remark: A function $f: \mathbb{N}^* \to \mathbb{Z}$ is called multiplicative if $f(mn) = f(m) f(n)$ whenever $\gcd(m,n) = 1$

The $\tau(\cdot)$ function is multiplicative. That is

$$\tau(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = \prod_{i=1}^{r} \tau(p_i^{\alpha_i})$$

Proof: (Induction on r) when $r = 0$ $\tau(n) = 1$

If $n = p_1^{\alpha_1}$, then the divisors of n are $1, p_1, p_1^2, \ldots p_1^{\alpha_1}$
$\implies \tau(n) = (\alpha_1 + 1)$. Assume the hypothesis true for $r-1$. Let $n = m p_r^{\alpha_r}$ where $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{r-1}^{\alpha_{r-1}}$

~~We may assume~~ $m < n$. From the hypothesis, the no: of divisors of m $\tau(m) = \prod_{i=1}^{r-1} (\alpha_i + 1)$

If $d \mid n$ then $d \mid m$ or $d \mid p_r^{\alpha_r}$. That is $d = d_1 d_2$
The no: of divisors of m are $\prod_{i=1}^{r-1} (\alpha_i + 1)$, and for $p_r^{\alpha_r}$ it is $(\alpha_r + 1)$. Therefore $\tau(n) = \tau(m)\tau(p_r^{\alpha_r}) = \prod_{i=1}^{r} (\alpha_i + 1)$

Example: $\tau(120) = (3+1)(1+1)(1+1) = 16$. $(120 = 2^3 \times 3 \times 5)$

Corollary: $p_1, p_2 \ldots p_r \in \mathbb{P}$, $\tau(p_1 p_2 \ldots p_r) = 2^r$

The product of divisors of n is denoted by $P(n) = n^{\tau(n)/2}$

Proof: List all the divisors of n.

$$1 = d_1 < d_2 < d_3 \ldots \quad < d_s = n \quad \text{where } s = \tau(n)$$

$d_1 d_s = n$

$d_2 d_{s-1} = n$

$\vdots \quad \vdots$

Case 1: S is even $= 2m$

$$d_m d_{m+1} = n \implies P(n) = n^m = n^{\tau(n)/2}$$

Case 2: S is odd $= 2m+1 \implies \alpha_i$ are even

Then $d_m d_{m+2} = n$, and $d_{m+1} = \sqrt{n} = p_1^{\alpha_1/2} p_2^{\alpha_2/2} \ldots p_r^{\alpha_r/2}$

(Note all α_i's are even). Therefore d_{m+1} is a integer.

$$P(n) = n^m \sqrt{n} = n^{\frac{2m+1}{2}} = n^{\tau(n)/2}$$

The sum of all divisors of n $\sigma(n)$ is

$$\sigma(n) = \prod_{i=1}^{r} \frac{(p_i^{\alpha_i+1} - 1)}{(p_i - 1)}$$

In particular, $\sigma(p^\alpha) = \dfrac{(p^{\alpha+1} - 1)}{(p-1)}$.

$$\& \ \sigma(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}) = \prod_{i=1}^{r} \sigma(p_i^{\alpha_i}).$$

Therefore $\sigma(n)$ is a multiplicative function.

Proof: Consider $p_1^{\mu_1} p_2^{\mu_2} \ldots p_r^{\alpha_r}$ where $0 \le \mu_i \le \alpha_i$, $i = 1, 2 \ldots$

$$\sigma(n) = \sum_{\substack{0 \le \mu_1 \le \alpha_1 \\ \vdots \\ 0 \le \mu_r \le \alpha_r}} p_1^{\mu_1} p_2^{\mu_2} \ldots p_r^{\mu_r}$$

$$\sigma(n) = \sum_{\substack{0 \leq \mu_1 \leq \alpha_1 \\ 0 \leq \mu_2 \leq \alpha_2 \\ \vdots \\ 0 \leq \mu_r \leq \alpha_r}} p_1^{\mu_1} p_2^{\mu_2} \cdots p_r^{\mu_r}$$

$$= \left(\sum_{0 \leq \mu_1 \leq \alpha_1} p_1^{\mu_1}\right)\left(\sum_{0 \leq \mu_2 \leq \alpha_2} p_2^{\mu_2}\right) \cdots \left(\sum_{0 \leq \mu_r \leq \alpha_r} p_r^{\alpha_r}\right)$$

$$= \prod_{i=1}^{r} \sum_{\mu_i} p_i^{\mu_i}$$

Note: $\sum_{\mu_i} p^{\mu_i}$ is a geometric series. Therefore

$$\sum_{\mu_i} p^{\mu_i} = 1 + p_i + p_i^2 \cdots p_i^{\alpha_i} = \frac{(p_i^{\alpha_i+1} - 1)}{(p_i - 1)}$$

Therefore, $\sigma(n) = \prod_{i=1}^{r} \frac{(p_i^{\alpha_i+1} - 1)}{(p_i - 1)}$

Another proof for $\tau(n)$: $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$.

~~$\tau(n) =$~~ All divisors of n are $p_1^{\mu_1} p_2^{\mu_2} \cdots p_r^{\mu_r}$

$0 \leq \mu_i \leq \alpha_i \;\; \forall\, i = 1, 2 \ldots r$. Therefore, the total number of divisors of n are $[0, \alpha_1] \times [0, \alpha_2] \times \cdots [0, \alpha_r]$

$[0, \alpha_i]$ denotes the No. of natural numbers in $[0, \alpha_i]$

$\Rightarrow \tau(n) = \prod_{i=1}^{r} (\alpha_i + 1)$.

Sylvester's Formula: Let $M_1, M_2, \ldots M_r$ be finite sets, then

$$(M_1 \cup M_2 \ldots \cup M_r) = \sum_{i=1}^{r} |M_i| - \sum_{1 \leq i_1 \leq i_2 \leq r} |M_{i_1} \cap M_{i_2}| \cdots$$

$$+ \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq r} |M_{i_1} \cap M_{i_2} \cap M_{i_3}| - \sum (-1)^{(r-1)} |M_1 \cap M_2 \cdots M_r|$$