27/9/05

What is the gcd of $\binom{n}{r}, \binom{n}{s}$?

Theorem: $n \in \mathbb{N}^*$, p prime number. Then

$$\vartheta_p(n!) = \sum_{i \geq 1}^{r} \left[\frac{n}{p^i}\right]$$

Proof: The sum on RHS is finite
Proof by induction on n.
Induction starts at ~~n=0~~. $n=1$.

Assume the result for $(n-1)$.

$$\vartheta_p((n-1)!) = \sum_{i \geq 1} \left[\frac{n-1}{p^i}\right]$$

Enough to prove that

$$\sum_{i \geq 1} \left[\frac{n}{p^i}\right] - \sum_{i \geq 1} \left[\frac{n-1}{p^i}\right] = j$$

$$\left[\frac{n}{p^i}\right] - \left[\frac{n-1}{p^i}\right] = \begin{cases} 1 & \text{if } p^i \mid n \\ 0 & \text{otherwise} \end{cases}$$

Observation: $a_1, a_2 \ldots a_n \in \mathbb{N}$

$f(1) = \#\{i \mid a_i \geq 1\}$

$f(2) = \#\{i \mid a_i \geq 2\}$

$f(n) = \#\{i \mid a_i \geq n\}$

$$a_1 + a_2 \dots a_n = f(1) + f(2) + f(3) \dots$$

~~Let $a_j = \vartheta_p(n)$~~

Let $a_j = \vartheta_p(\overset{n}{\cancel{a_j}})$ $1 \le j \le n$

$$f(j) = \# \{ m \mid m \le n, p \mid m \} =$$
$$= p, 2p, \dots \left[\frac{n}{p^i}\right] p = \left[\frac{n}{p}\right]$$

$$f(2) = \# \{ \dots \mid p^2 \mid m \}$$
$$= \left[\frac{n}{p^2}\right]$$

$$f(k) = \# \{ \dots \mid p^k \mid m \}$$
$$= \left[\frac{n}{p^k}\right]$$

From the above observation, the result follows

Cor 1 $a_1, a_2, \dots a_r \in \mathbb{N}$

if $n = a_1 + a_2 \dots a_r$ Then $\dfrac{n!}{a_1!\, a_2! \dots a_r!} \in \mathbb{Z}$

We will check by showing that $\vartheta_p(\cdot)$ is ≥ 0.

$$\vartheta_p(n!) - \vartheta_p(a_1!\, a_2! \dots a_r!)$$

Enough to prove that

$$\Rightarrow \vartheta_p(n!) - \left[\vartheta_p(a_1!) + \dots \vartheta_p(a_r!)\right] \ge 0$$

$$\frac{a_1}{p^i} + \frac{a_2}{p^i} \dots \frac{a_n}{p^i} = \frac{n}{p^i}$$

$$\Rightarrow \left[\frac{a_1}{p^i}\right] + \left[\frac{a_2}{p^i}\right] \dots \left[\frac{a_n}{p^i}\right] \le \left[\frac{a_1 + a_2 \dots a_n}{p^i}\right]$$

$$= \cancel{\sum_{i \geq 1} \left[\frac{n}{p_i}\right]}$$

$$\sum_{j=1}^{r} \sum_{i \geq 1} \left[\frac{a_j}{p^i}\right] \leq \sum_{i \geq 1} \left[\frac{n}{p^i}\right] = \nu_p(n!)$$

Cor 2: If $a_1, a_2 \ldots a_k$ are consecutive natural numbers. The $k!$ divides $a_1 \ldots a_k$

Proof: wma

$$\underset{\Downarrow}{a_1} < a_2 \ldots a_k =: n.$$

$$n-k+1$$

$$\frac{\lfloor n}{\lfloor n-k} = n(n-1) \ldots (n-k+1) = a_1 a_2 \ldots a_k$$

Etpt $k! \,\Big|\, \frac{n!}{n-k!}$

i.e. $\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \in \mathbb{Z}$ by corollary 1.

Show that $a, b \in \mathbb{N}$ $\frac{(ab)!}{a!\,(b!)^a} \in \mathbb{Z}$

How to compute $100!$

$$100! = (1.3.5 \ldots\ldots 99) \underbrace{(2.4.6 \ldots\ldots 98, 100)}$$

$$= 2^{50}(1.3.5. \ldots\ldots 50)$$

$$100! = 2^{50} \cdot 50! \ (1.3.5 \ldots 99)$$

List the primes till 50.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

$n \in \mathbb{N}$ $p \in \mathbb{P}$

$$\sum_{i \geq 1} \left[\frac{n}{p^i}\right] = \left[\frac{n - (a_0 + a_1 + a_2 \ldots a_r)}{(p-1)}\right]$$

Using the fact

$$n = a_r p^r + a_{r-1} p^{r-1} \ldots + a_0$$

$$(a_r, a_{r-1} \ldots a_0)_p$$

p-adic expansion

$x \in \mathbb{R}$, $\pi(x) = \# \{p \in \mathbb{P} \mid p \leq x\}$.

That is the no: of primes less than x.

Legendre's formula:

$N \in \mathbb{N}^*$, $P_1, P_2, \ldots P_r$ distinct primes

$P_1, P_2, P_3 \ldots P_r \leq \sqrt{N}$, $r = \pi(\sqrt{N})$

$$\pi(N) = N + r - 1 - \sum_{i_1} \left[\frac{N}{P_{i_1}}\right] + \sum_{i_1 < i_2} \left[\frac{N}{P_{i_1} P_{i_2}}\right] \ldots + \cancel{(-1)}$$

$$\ldots (-1)^r \left[\frac{N}{P_1 \ldots P_r}\right]$$