Examples of category

1. $(A, \leq)$ ordered set

(a) objects are elements of A

(b) For any $x, y \in A$, define

$$\mathrm{Hom}_A(x,y) = \begin{cases} \{*\} & x \leq y \\ \phi & \text{otherwise} \end{cases}$$

Note that elements of $\mathrm{Hom}_A(x,y)$ are not ~~ea~~ maps. It is either $\{*\}$ or empty.

For $x, y, z \in A$,

if $\mathrm{Hom}_A(x,y) = \phi$ or $\mathrm{Hom}_A(y,z) = \phi$

then $\phi \times \mathrm{Hom}_A(\ ) \longrightarrow \phi$

else

$$x \leq y,\ y \leq z \implies x \leq z$$

(Due to the property of ordered set).

~~He~~

(c) $\mathrm{Hom}_A(x,x) = \{*_{xx}\}$

~~If we~~ The above element is the identity which can be trivially verified.

Also,

$$\mathrm{Hom}_A(x,x) \times \mathrm{Hom}_A(x,y) \longrightarrow \mathrm{Hom}_A(x,y)$$

$$(\{*_{xx}\} \times \{*_{xy}\}) \longrightarrow \{*_{xy}\}$$

2. X is any set, τ is an topology on X if

$\tau \subseteq P(X)$ Power set of X satisfying

(i) $\phi, X \in \tau$

(ii) τ is closed under intersection ~~$u, v \in \tau$~~

$u, v \in \tau \implies u \cap v \in \tau$

(iii) τ is closed under arbitrary union

$U_i \in \tau,\ i \in I \implies \bigcup_{i \in I} U_i \in \tau.$

The power set is a topology on X which is called discrete topology.
Also $\tau = \{\emptyset, X\}$ is also a topology. (indiscrete topology).
The elements of τ are called open subsets ~~τ topology~~ in X.

Fix a topology on τ. To show τ is a category.
(a) objects are elements of τ. (subsets of X)
(b) For $U, V \in \tau$

$$\mathrm{Hom}_\tau(U,V) = \begin{cases} \{U \hookrightarrow V\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

$\{U \hookrightarrow V\}$ is the natural inclusion map.
Here $\mathrm{Hom}_\tau(U,V)$ ~~is~~ ~~a~~ contains a map.

For any $U, V, W \in \tau$

$$\mathrm{Hom}_\tau(U,V) \times \mathrm{Hom}_\tau(V,W) \longrightarrow \mathrm{Hom}_\tau(U,W)$$
$$U \subseteq V \qquad V \subseteq W \implies U \subseteq W$$

Rings: $(R, +, \cdot)$; $+$ is addition on R
$\cdot$ is ~~add~~ multiplication on R.
R is a group.
(i) $(R,+)$ is an abelian group
$x \in R$, $-x \in R$, $x + (-x) = 0 = (-x) + x$
0 is the additive identity
(ii) $(R, \cdot)$ is a monoid
identity is denoted by 1. (also called the multiplicative identity).
Note $(R,\cdot)$ need not be commutative and neither cancellative

(iii) ($+$ and $\cdot$) are distributive

$$x\cdot(y+z) = x\cdot y + y\cdot z$$
$$(y+z)\cdot x = y\cdot x + z\cdot x$$

Examples of rings: $(\mathbb{Z},+,\cdot)$, $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$

The following properties can be shown to be true.

1) $0\cdot x = x\cdot 0 = 0$
2) $x(-y) = -x\cdot y = (-x)\cdot y$
3) $(-x)(-y) = x\cdot y$
4) $m\in\mathbb{N}$, $mx = \underbrace{x + x + x}_{m \text{ times}}$

   $m(x+y) = mx + my$

   If $m<0$ $\quad mx = \underbrace{(-x)+(-x)\cdots(-x)}_{m \text{ times}}$

5) $(mx)(ny) = (mn)(xy)$ This is true irrespective of the commutativity.

Ring Homomorphisms: R and R' are rings

A ring homomorphism has the following property

$R \xrightarrow{f} R'$

- $(R,+) \xrightarrow{f} (R',+')$ is a group homomorphism.
  $\Rightarrow f(x+y) = f(x)+f(y)$
  $f(0) = 0$
- $(R,\cdot) \xrightarrow{f} (R',\cdot')$ is a monoid homomorphism.
  $\Rightarrow f(1) = 1$, $f(xy) = f(x)\,f(y)$

Rings form a category.

$\mathrm{Hom}_{\mathrm{Rings}}(\mathbb{Z}, R)$ (any ring R).

$\mathbb{Z} \xrightarrow{\gamma_R} R \Rightarrow 1 \longmapsto 1_R$. There is exactly one ring homomorphism from $\mathbb{Z}$ to any ring.