16/3.

Generally, If $R$ is a Ring, $a, b \in R$, char $R = p$, $p \in \mathbb{P}$, $n \in \mathbb{N}^*$, then $(a+b)^{p^n} = a^{p^n} + b^{p^n}$ ~~pro~~.

If $K$ is a field, and char $K = 0 \Rightarrow \mathbb{Q} \subseteq K$.
If char $K > 0 \Rightarrow$ char $K$ is prime. ~~$\mathbb{Z}$~~
Also $K$ contains $\mathbb{Z}_p$ (prime ring of $K$). $\mathbb{Z}_p$ is also a field.
Therefore $\mathbb{Q}$, and $\mathbb{Z}_p, p \in \mathbb{P}$ are the only prime fields.

## $\boxed{\text{Modules \& Algebras}}$

17/1.

Module: Let $R$ be a ring. $V$ is a $R$-module if
(a) $(V, +)$ is a abelian group
(b) There is a scalar multiplication of $R$ on $V$ such that: $R \times V \longrightarrow V$
$$(a, x) \longrightarrow ax$$
(c) The scalar multiplication has the following properties
(i) $a(bx) = (ab)x$ (associative)
(ii) $(a+b)x = ax + bx$ (distributive)
(iii) $a(x+y) = a \cdot x + b \cdot y$
(iv) $1 \cdot x = x$

Examples: 1. An ring ~~is~~ $R$ is an $R$-module.
2. An ideal $A$ in $R$ is a $R$-module.
3. $V$ is any $R$-module, $I$ any set
$$V^I = \{f : I \to V\}, \quad I \text{ tuples in } V.$$
Show that this is a $R$-module, with the operations
$(f+g)(i) = f(i) + g(i)$, $(af)(i) = a f(i)$, $f, g \in V^I$, $a \in R$.
Verify that $V^{(I)}$ is also a module (a sub-module of $V^I$)

~~$R$ is a submodule of a $V$ module $V$, if $R \subseteq V$~~

& W is a R-submodule of a R-module V, if $W \subseteq V$ and

$$W \begin{cases} (W,+) \subseteq (V,+) \\ R \times W \to W \end{cases}$$

Examples: (i) A ideal A in a ring R is a R-submodule of R.

(ii) Let I be any set, $V_i$, $i \in I$ be a family of modules, $\prod_{i \in I} V_i$ ~~is~~ (direct product of $V_i$, $i \in I$).

Show that the direct sum $\coprod_{i \in I} V_i$ is a R-submodule of the direct product.

Particular case: ~~$V_1, V_2, V_3 \ldots V_n$~~ Let $V = R^I$, $R^{(I)} \subseteq R^I$

Define the standard basis $e_i = (\delta_{ij})_{j \in I}$

$$\updownarrow$$

$$e_i : I \to R$$
$$j \mapsto \delta_{ij}$$

Any element in $R^{(I)}$ can be uniquely written as $x = \sum_{i \in I} a_i e_i$ where $(a_i) \in R^{(I)}$.

$R^{(I)}$ is a free R-module with basis $e_i$, $i \in I$.

Abelian groups is $\mathbb{Z}$-module.

<u>Algebras</u>: Let R be a ring. A R-module A is a algebra if

(i) It is a ring with ~~s~~ operations $+_A$, $\cdot_A$.

(ii) There is a scalar multiplication $R \times A \to A$

$$(a, x) \mapsto ax.$$
$$(b, x) \mapsto bx$$

Such that (a) $(ax) \cdot (by) = (ab)(x \cdot y)$ $\forall a, b \in R, \forall x, y \in A$.

Examples: 1. $A = M_n(R)$ (set of $n\times n$ ~~aa~~ real matrices) $\downarrow R^{\{1,2\dots n\}\times\{1,2\dots n\}}$

This is a ring with the usual addition ~~am~~ and multiplication of matrices.

~~Note this is algebra~~ The scalar multiplication makes $M_n(R)$ an algebra. Note that this R-algebra is not commutative

2: Note that $R^I$ is a R-algebra, where as $R^{(I)}$ ~~w~~ is not an R-algebra.

<u>Another definition of R-algebra</u>: An R-algebra is a pair $(A, \phi)$ where $A$ is a ring and $\phi : R \longrightarrow A$ is a ring homomorphism such that $\phi(R) \subseteq Z(A)$ Centre of $A$ = ~~$\{a\in A|$~~

$= \{a\in A \mid ax = ax \ \forall x\in A\}$

$\phi$ is called the structure homomorphism of the R-algebra. ~~Show~~ Define $R\times A \to A$

$(a, x) \longmapsto \phi(a).x.$

Show that using the above scalar multiplication A is a R-module. Also, verify that this definition is the same as the previous one.

In the examples below, the base ring is $\mathbb{R}$.

(a) $A = \mathbb{R}^{[a,b]}$ (set of ~~cto~~ real-valued functions defined on the closed interval $[a,b]$.)

A is a R-algebra

(b) $C_{\mathbb{R}}([a,b]) = \{f\in\mathbb{R}^{[a,b]} \mid f \text{ is continuous}\}$

$C^0_{\mathbb{R}} \subseteq \mathbb{R}^{[a,b]}$, and $C^0_{\mathbb{R}}$ is a R sub-algebra

(c) $C^k_{\mathbb{R}}[[a,b])$ = Set of functions defined on the interval $[a,b]$ which are at least k times differentiable.

(d) $C^{\infty}_{\mathbb{R}}([a,b]) = \bigcap_{k \in \mathbb{N}} C^{k}_{\mathbb{R}}([a,b])$

R- algebra of infinitely differentiable functions.

(e) R is a (base ring), a family of R-algebras. $B_i$, $i \in I$, $\bigcap_{i \in I} B_i$ is also a R-subalgebra

(f) $\mathbb{C}$ is ~~a~~ the base ring $D = B(0,1)$ ball of radius 1.

$$\mathbb{C}^{1}_{\mathbb{C}}(D) = \mathbb{C}^{2}_{\mathbb{C}}(D) = \ldots \quad \mathbb{C}^{20}_{\mathbb{C}}(D)$$

↓ Analytic functions

Once differentiable functions are ~~ana~~ analytic functions