Polynomial Rings

Let A be a commutative ring. $A[\mathbb{N}^{(I)}]$ is the monoid ring $\mathbb{N}^{(I)}$ over A. Any $f \in A[\mathbb{N}^{(I)}]$ can be written as $f = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu e_\nu$.

Since $\nu \in \mathbb{N}^{(I)}$ can be expressed as $\nu = \sum_{i \in I} \nu_i e_{\varepsilon_i}$, the multiplication operation defined on $A[\mathbb{N}^{(I)}]$ implies

$$f = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu \prod_{i \in I} (e_{\varepsilon_i})^{\nu_i} = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X_i^{\nu_i}$$

~~The~~ $A[\mathbb{N}^{(I)}]$ is also called the polynomial ring over A, and is denoted by $A[X_i \mid i \in I]$

<u>Substitution homomorphism</u>: Let B be an A-algebra, $b_i \in B$, $i \in I$ (any set). Then $\exists$ a unique A-algebra homomorphism from $A[X_i \mid i \in I] \longrightarrow B$ such that $X_i \longmapsto b_i$

Proof: Construct a monoid homorphism α

$$\alpha : \mathbb{N}^{(I)} \longrightarrow B$$

$$\sum_{i \in I} \nu_i \varepsilon_i \longmapsto \sum_{i \in I} (b_i)^{\nu_i} = b^\nu .$$

Note ~~(ti)~~ (ν_i) is finitely many non-zero.

The above map take ε_i to b_i. Show that α is a monoid homomorphism. We ~~now~~ invoke the universal property, and we have a unique homomorphism

$$\psi : A[\mathbb{N}^{(I)}] \longrightarrow B$$

$$f = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X^\nu \longrightarrow \sum a_\nu b^\nu$$

$$\begin{tikzcd} \sigma, \varepsilon_i \in \mathbb{N}^{(I)} \arrow[r] \arrow[d] & B \ni e_{\alpha(\sigma)}, b_i \\ e_{\varepsilon_i}, e_\sigma \in A[\mathbb{N}^{(I)}] \arrow[ur] & \end{tikzcd}$$

One can think of $f = \sum_{\gamma} a_\gamma X^\gamma \longrightarrow \sum_{\gamma} a_\gamma b^\gamma = f(b)$

Example: $I = \{1, 2, \dots n\}$, $\cancel{\nu = \{1, 2,}$ Any $\nu \in \mathbb{N}^I$ is of the form $\nu = \{\nu_1, \nu_2, \dots \nu_n\}$.

Any $X^\nu \in A[\mathbb{N}^n]$ is of the form $X^\nu = X_1^{\nu_1} \dots X_n^{\nu_n}$,

where $X_i = e_{\varepsilon_i}$ $i \in I$ $\varepsilon_i = (0, \dots \underset{i^{th}\text{ position}}{1} \dots 0)$

$A[\mathbb{N}^n] = A[X_1, X_2 \dots X_n]$. Any $f \in A[\mathbb{N}^n]$ can be expressed in $\{X_i\}_{i \in I}$ as

$$f = \sum_{\nu \in \mathbb{N}^I} a_\nu X^\nu, \text{ where } a_\nu \in A^{(\mathbb{N}^I)}$$

Any $f, g \in A[\mathbb{N}^I]$, $f = \sum_{\nu} a_\nu X^\nu$, $g = \sum b_\nu X^\nu$

$$\Longrightarrow f + g = \sum_{\nu} (a_\nu + b_\nu) X^\nu, \text{ and}$$

$$fg = \sum_{\nu = \cancel{\lambda+\mu}} c_\nu X^\nu, \text{ where } c_\nu = \sum_{\lambda + \mu = \nu} a_\lambda b_\mu$$

Let $b_1, b_2 \dots b_n \in B$. By the map $X_i \longrightarrow b_i$, $i \in I$, verify (i) $f(b) + g(b) = (f+g)b$, (ii) $f(b)\,g(b) = (fg)(b)$.

Let B be an A-algebra, x_i, $i \in I$ in B. Denote $\underline{x} = (x_i)_{i \in I}$

$$Q_{\underline{x}} : A[X_i \mid i \in I] \longrightarrow B$$

$$X_i \longrightarrow \cancel{b_i}\, x_i$$

$$f \longmapsto \sum_{\nu} a_\nu \prod_{i \in I} x_i^{\nu_i} = f(\underline{x})$$

That is $Q_{\underline{x}}(f) = f(\underline{x})$.

Verify that $\operatorname{Im} Q_{\underline{x}}$ is a subring of B.

Image Q_x is also denoted as $A[x_i \mid i \in I]$. Note that the expression for $f(\underline{x}) = \sum_{\nu} a_\nu x^\nu$ is no longer unique. $f(\underline{x})$ is called the value of f at $\underline{x}$.

$\underline{x}$ is called a zero of f if $f(\underline{x}) = 0$.

Example: $I = \{1\}$, $A = \mathbb{Z}$, monoid $\mathbb{N}$, the polynomial ring $\mathbb{Z}[X]$. Let $B = \mathbb{Q}$, and $x = 1/2$.

$$\mathbb{Z}[X] \xrightarrow{Q_{1/2}} B$$
$$X \longrightarrow 1/2$$ Even though $f(X)$ is unique
$$f(X) \longmapsto f(1/2)$$ $f(\frac{1}{2})$ is not unique

$$f(0) \longmapsto 0$$
$$2X \longmapsto \left(2\cdot\tfrac{1}{2}\right) = 1$$
$$2X-1 \longmapsto \left(2\cdot\tfrac{1}{2}-1\right) = 0 \Rightarrow \tfrac{1}{2} \text{ is a zero for } 2X-1$$

More examples in another class.