

POLYNOMIAL RINGS

L19/11

Let A be a commutative ring. $A[N^{(I)}]$ is the monoid ring $N^{(I)}$ over A . Any $f \in A[N^{(I)}]$ can be written as $f = \sum_{\nu \in N^{(I)}} a_{\nu} e_{\nu}$.

Since $\nu \in N^{(I)}$ can be expressed as $\nu = \sum_{i \in I} \nu_i e_{\xi_i}$,

the multiplication operation defined on $A[N^{(I)}]$ implies

$$f = \sum_{\nu \in N^{(I)}} a_{\nu} \prod_{i \in I} (e_{\xi_i})^{\nu_i} = \sum_{\nu \in N^{(I)}} a_{\nu} X_i^{\nu_i}$$

~~The~~ $A[N^{(I)}]$ is also called the polynomial ring over A , and is denoted by $A[X_i | i \in I]$.

Substitution homomorphism: Let B be an A -algebra, $b_i \in B$, $i \in I$ (any set). Then \exists a unique A -algebra homomorphism from $A[X_i | i \in I] \longrightarrow B$

such that $X_i \longmapsto b_i$

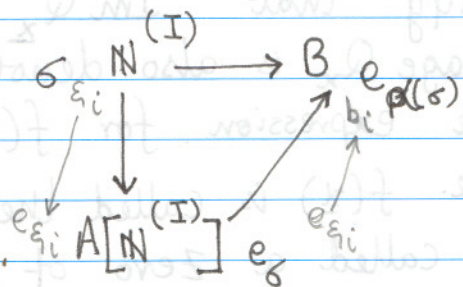
Proof: Construct a monoid homomorphism α
 $\alpha: N^{(I)} \longrightarrow B$

$$\sum_{i \in I} \nu_i \xi_i \longmapsto \sum_{i \in I} (b_i)^{\nu_i} = b. \quad \text{Note that } (\nu_i) \text{ is finitely many non-zero.}$$

The above map takes ξ_i to b_i . Show that α is a monoid homomorphism. We now invoke the universal property, and we have a unique homomorphism

$$\psi: A[N^{(I)}] \longrightarrow B$$

$$f = \sum_{\nu \in N^{(I)}} a_{\nu} X^{\nu} \longrightarrow \sum a_{\nu} b^{\nu}$$



One can think of $f = \sum_{\gamma} a_{\gamma} X^{\gamma} \rightarrow \sum_{\gamma} a_{\gamma} b^{\gamma} = f(b)$

Example: $I = \{1, 2, \dots, n\}$, ~~$\gamma = \{1, 2, \dots, n\}$~~ Any $\gamma \in \mathbb{N}^I$ is of the form $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$.

Any $X^{\gamma} \in A[N^n]$ is of the form $X^{\gamma} = X_1^{\gamma_1} \dots X_n^{\gamma_n}$,

where $X_i = e_{\xi_i}$ $i \in I$ $\xi_i = (0, \dots, 1 \dots 0)$
 i^{th} position

$A[N^n] = A[X_1, X_2, \dots, X_n]$. Any $f \in A[N^n]$ can be expressed in $\{X_i\}_{i \in I}$ as

$$f = \sum_{\gamma \in \mathbb{N}^I} a_{\gamma} X^{\gamma}, \text{ where } a_{\gamma} \in A^{(\mathbb{N}^I)}$$

Any $f, g \in A[N^I]$, $f = \sum_{\gamma} a_{\gamma} X^{\gamma}$, $g = \sum_{\gamma} b_{\gamma} X^{\gamma}$

$$\Rightarrow f+g = \sum_{\gamma} (a_{\gamma} + b_{\gamma}) X^{\gamma}, \text{ and}$$

$$fg = \sum_{\gamma = \lambda + \mu} c_{\gamma} X^{\gamma}, \text{ where } c_{\gamma} = \sum_{\lambda + \mu = \gamma} a_{\lambda} b_{\mu}$$

let $b_1, b_2, \dots, b_n \in B$. By the map $X_i \rightarrow b_i$, $i \in I$, verify (i) $f(b) + g(b) = (f+g)(b)$, (ii) $f(b)g(b) = (fg)(b)$.

let B be an A -algebra, x_i , $i \in I$ in B . Denote $\underline{x} = (x_i)_{i \in I}$

$$Q_{\underline{x}}: A[x_i | i \in I] \rightarrow B$$

$$x_i \mapsto b_i x_i$$

$$f \mapsto \sum_{\gamma} a_{\gamma} \prod_{i \in I} x_i^{\gamma_i} = f(\underline{x})$$

That is $Q_{\underline{x}}(f) = f(\underline{x})$.

Verify that $\text{Im } Q_{\underline{x}}$ is a subring of B .

Image $Q_{\underline{x}}$ is also denoted as $A[x_i | i \in I]$. Note that the expression for $f(\underline{x}) = \sum_{\gamma} a_{\gamma} x^{\gamma}$ is no longer

unique. $f(\underline{x})$ is called the value of f at \underline{x} .

\underline{x} is called a zero of f if $f(\underline{x}) = 0$.

Example: $I = \{1\}$, $A = \mathbb{Z}$, monoid \mathbb{N} , the polynomial ring $\mathbb{Z}[X]$. let $B = \mathbb{Q}$, and $x = 1/2$.

$$\mathbb{Z}[X] \xrightarrow{\mathbb{Q}_{1/2}} B$$

$$X \longrightarrow 1/2$$

$$f(X) \longmapsto f(1/2)$$

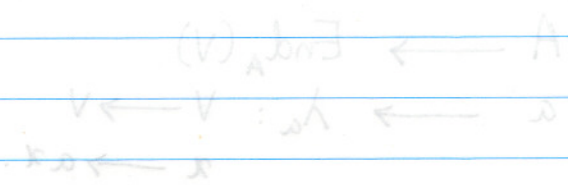
$$f(0) \longmapsto 0$$

$$2X \longmapsto (2 \cdot 1/2) = 1$$

$$2X - 1 \longmapsto (2 \cdot 1/2 - 1) = 0 \implies 1/2 \text{ is a zero for } 2X - 1$$

Even though $f(X)$ is unique $f(1/2)$ is not-unique

More examples in another class.



Verify that $\text{End}_A(V)$ is a A -algebra by checking the following:

- (i) $\lambda + \mu = \lambda + \mu$
- (ii) $\lambda \cdot \mu = \mu \cdot \lambda$
- (iii) $\lambda \cdot 1 = \lambda$
- (iv) $\lambda \cdot \mu = \mu \cdot \lambda \quad \forall \lambda, \mu \in \text{End}_A(V)$

The structure ring homomorphism from A to $\text{End}_A(V)$ is λ .

let $A = K$ (a field), V be a finite dimensional vector space over K

$$\text{End}_K(V) \cong M_n(K) \text{ (K-algebra)}$$