# Modules



$V$ is a $A$-module if (i) $(V,+)$ is a Abelian group,
(ii) there is a scalar multiplication $A \times V \to V$, $(a,x) \to ax$ with the associative and distributive properties.
Also, $(1,x) = 1\cdot x = x$.

$A$-submodule of $V$: It is a subgroup of $(V,+)$ and a subset of $V$ closed under the scalar multiplication.

Let $V, W$ be $A$-modules. $f: V \to W$ is a module homomorphism if (i) $f(x+y) = f(x) + f(y)$, (ii) $f(ax) = af(x)$
Note that $W^V$ is a ~~sub~~-$A$-module, and $\mathrm{Hom}_A(V,W) \subseteq W^V$.
The set of all module homomorphisms from $V \to W$
$\mathrm{Hom}_A(V,W)$ is a $A$-submodule of $W^V$.

Verify that set of $A$-modules is a category.

$\mathrm{Hom}_A(V,V) = \mathrm{End}_A(V)$. It is a ring with composition as the multiplication. (Obviously, not commutative)

$$\begin{aligned} A &\longrightarrow \mathrm{End}_A(V) \\ a &\longrightarrow \lambda_a : V \longrightarrow V \\ &\qquad\quad x \longrightarrow ax. \end{aligned}$$

Verify that $\mathrm{End}_A(V)$ is a $A$-algebra by checking the following:
(i) $\lambda_{a+b} = \lambda_a + \lambda_b$
(ii) $\lambda_{ab} = \lambda_{(a)} \circ \lambda_{(b)}$
(iii) $\lambda_1 = id_V$
(iv) $\lambda_a \circ f = f \circ \lambda_a \quad \forall a \in A, \forall f \in \mathrm{End}_A V$.

The structure ring homomorphism from $A$ to $\mathrm{End}_A(V)$ is $\lambda$.

Let $A = K$ (a field), $V$ be a finite dimensional vector space over $K$.

$$\mathrm{End}_K(V) \cong M_n(K) \quad (K\text{-algebra homomorphism})$$
$$f \longmapsto M_{\underline{V}}^{\underline{V}} f$$

Let $A$ be a ring and $x_i$, $i \in I$ be a family of elements in $V$. The set $\{x_i\}_{i \in I}$ is called a <u>generating set</u> if every element $x \in V$ can be expressed as $x = \sum_{i \in I} a_i x_i$, $(a_i) \in A^{(I)}$

<u>Linear Independence</u>: The set $\{x_i\}_{i \in I}$ is linearly independent over $A$ if $\sum_{i \in I} a_i x_i = 0 \Rightarrow a_i = 0 \ \forall i \in I$

<u>Illustrative Example</u>: Let $A = \mathbb{Z}$. The set $\{2, 3\}$ is a generating set, or any $(a, b)$ such that $\gcd(a, b) = 1$ generates the module $\mathbb{Z}$. However, this set is not linearly independent $(2 \cdot 3 - 3 \cdot 2 = 0)$.

$\{x_i\}_{i \in I}$ is called a minimal generating set for $V$ if $\forall j \in I$ $\{x_i\}$ $i \in I \setminus \{j\}$ is not a generating set of $V$.

In the above example $\{2, 3\}$ is a minimal generating set but it is not linearly independent.

However, for a vectorspace, any minimal generating set is a basis. For $\mathbb{Z}$, $\{1\}$ is a $A$-basis.

<u>Basis</u>: $\{x_i\}$ $i \in I$ is a $A$-basis of $V$ if it is a generating set and if it is linearly independent over $A$.

<u>Example of a module not having a basis</u>:

Let $A = \mathbb{Z}$, module $V = \mathbb{Z}_2$ ($\mathbb{Z}$ mod 2). $V = \{0, 1\}$.

$\mathbb{Z}_2$ is a $\mathbb{Z}$-module. Show that any Abelian group is a $\mathbb{Z}$-module $n \cdot x = \underbrace{x + x + \cdots x}_{n \text{ times}}$

For $\mathbb{Z}_2 = \{0, 1\}$, $\{1\}$ is a minimal generating set. But $4 \cdot 1 = 2 \cdot 1 = 0$. Hence $\{1\}$ is not linearly independent. Therefore, $\mathbb{Z}_2$ has no basis.

Note that $\mathbb{Z}_n$ or $\mathbb{Z}$-modules generated from any finite Abelian group does not have basis.

However, $A$ as a $A$-module always has a basis namely $\{1\}$ or any $x \in A^{\times}$. Verify that if $\{a\}$ is a basis of $A$ as a $A$-module, then $a \in A^{\times}$. It is clear that basis cannot have zero.

Let $V$ be a $A$-module and $\{x_i\}$ $i \in I$ be elements in $V$. The smallest $A$-submodule containing $x_i$, $i \in I$ is precisely

$$\left\{ \sum_{i \in I} a_i x_i \,\middle|\, (a_i) \in A^{(I)} \right\} := \sum_{i \in I} A x_i$$

This sub-module is generated by the elements $x_i$

Consider the module $A^{(I)}$. Any element $a \in A^{(I)}$ can be written as $a = \sum_{i \in I} a_i e_i$ ($e_i$ is the standard basis).

$\{e_i\}$ $i \in I$ generates $A^{(I)}$ and also

$$\sum_i \lambda_i e_i = 0 \implies \left(\sum_i \lambda_i e_i\right)(j) = 0 \implies \lambda_j = 0 \ \forall j \in I$$

Therefore, $\{e_i\}_{i \in I}$ is a basis.

<u>Free Module:</u> An $A$-module $V$ is called free over $A$ if $V$ has a basis.

If $A = K$ (a field), then every $K$-module (i.e. $K$-vector space) has a basis, and hence free.