# FREE MODULES



Let A be a ring. If A is commutative, then any two basis of ~~any~~ a A-module will have the same cardinality. (Proof later).

**Rank:**

A be a ring (may not be commutative), V is a A-module if V has a basis (may not always be true), then we say that V has a rank if any two basis of V have the same cardinality. This cardinality is called the rank of V (denoted by $\text{rank}_A V$).

21.1 **Theorem:** Suppose V is not finitely generated, i.e. V ~~is~~ has an infinite basis, $x_i, i \in I$. Let $y_j, j \in J$ be another basis of V. Then $|I| = |J|$ i.e. V has a rank. (Note A may not be commutative). (Proof later)

21.2 **Theorem:** If (i) A is commutative, (ii) A is non-zero finite ring (iii) A is a division ring, and A is finitely generated, i.e. V has a finite basis, $\{x_i\}_{i \in I}$, $I = \{1, 2, \ldots n\}$ then every basis has a cardinality n. The module V has a rank n.

Example: We will construct a module ~~whose~~ where the cardinality of ~~any~~ two basis is different.

Let A be a non-commutative ring which is non-zero. $V = A$. Therefore, $\{1\}$ as a basis exists. Let $x, y \in A$, when does $x, y$ become a basis of V?

If $\{x, y\}$ generates $V \iff 1 = ax + by$ for some $a, b \in A$.

To check for linear independence:

$$x = (x\cdot a)\cdot x + x(by)\cdot \cancel{x}$$

$$0 = (1 - x\cdot a)x + (x\cdot b)y \Rightarrow x\cdot a = 1, \; \& \; x\cdot b = 0$$

Similarly $y = ya\cdot x + y\cdot b\cdot y \Rightarrow 0 = (y\cdot a)x + (y\cdot b - 1)y$
$\Rightarrow y\cdot a = 0 \;\&\; y\cdot b = 1.$

For linear independence: (i) $x\cdot a = 1$, (ii) $y\cdot a = 0$,
(iii) $x\cdot b = 0$ (iv) $y\cdot b = 1$

<u>proof</u>:

if $\lambda x + \mu y = 0, \Rightarrow \lambda x\cdot a + \mu\cdot y\cdot a = 0$
$\Rightarrow \lambda = 0.$

~~Simildl~~ Similarly $\lambda\cdot x\cdot b + \mu\cdot y\cdot b = 0 \Rightarrow \mu = 0.$

Therefore, it is necessary and sufficient for linear independence that the four conditions be satisfied.

21.1 Proposition: Let $A$ be a non-commutative ring, $a, b, x, y \in A$ such that (i) $ax + by = 1$, (ii) $x\cdot a = 1$, (iii) $x\cdot b = 0$, (iv) $y\cdot a = 0$, (v) $y\cdot b = 1$. Then for any $n \in \mathbb{N}^*$, $y, yx, yx^2, \ldots yx^{n-1}, yx^n$ is a basis of module $V = A$ over $A$.

Proof: Let $n \geq 2$. Consider the equation (condition (i))

$$1 = ax + b\cdot y \longrightarrow (1)$$

Multiply (1) by $a$ on the left and by $x$ on the right.

We get, $ax = a^2x^2 + a\cdot b\, yx \longrightarrow (2)$

Multiply (2) by $a$ on the left and by $x$ on the right to get $a^2x^2 = a^3x^3 + a^2 b\, yx^2 \longrightarrow (3)$

Repeat this process till we get

$$a^{n-1}x^{n-1} = a^n x^n + a^{n-1} b y x^{n-1} \longrightarrow (n)$$

Adding (1), (2) . . . (n), we obtain

$$1 = \cancel{ax} + (a\cdot b)yx + (a^2 b)yx^2 \cdots (a^{n-1}b)yx^{n-1} + a^n x^n$$

$$1 = \cancel{ax} + (a\cdot b)yx + (a^2b)yx^2 \cdots (a^{n-1}b)yx^{n-1} + a^n x^n$$

Since $1$ can be obtained by a finite linear combination of the set ~~$\{ax$~~ $\{\cancel{x,}\ yx, yx^2 \ldots\ldots yx^{n-1}, x^n\}$, it constitutes a generating set.

To establish linear independence, consider the equation

$$0 = \lambda_1 yx + \lambda_2 yx^2 + \cdots \lambda_{n-1}yx^{n-1} + \lambda_n x^n$$

Multiply $(a\cdot b)$ on the right side of the above equation

$$0 = \lambda_1 yx\cdot a\cdot b + \lambda_2 yx^2 a\cdot b \;\; \cdot \;\; \lambda_{n-1} yx^{n-1}a\cdot b + \lambda_n x^n a\cdot b$$

$\Rightarrow\ 0 = \lambda_1$ (because $x\cdot a = 1$, $y\cdot b = 1$, $x\cdot b = 0$)

$\rightarrow$ Multiply $(a^2\cdot b)$ on the right side to get $\lambda_2 = 0$.

Proceeding similarly show that $\lambda_3 = \lambda_4 = \ldots \lambda_n = 0$.

Therefore the set $\{yx, yx^2, \ldots yx^{n-1}, x^n\}$ is a basis.

Example of a ring $A$ satisifying the five conditions in the above proposition.

Let $K$ be any field. The direct sum $K^{(\mathbb{N})}$ is a ~~ring with the operations +, and~~ vector space with the standard basis $\{e_n\}$ $n \in \mathbb{N}$. Denote $\mathbb{N}_0$ – the set of even natural numbers, $\mathbb{N}_1$ – the set of odd natural numbers. Therefore $\mathbb{N} = \mathbb{N}_0 \uplus \mathbb{N}_1$ (disjoint union).

There exists bijective maps $\sigma_0 : \mathbb{N} \longrightarrow \mathbb{N}_0$, $\sigma_1 : \mathbb{N} \longrightarrow \mathbb{N}_1$

Consider the ring $A = \mathrm{End}_K(V)$: $K$ linear maps can be completely specified by giving their values on ~~t~~ a basis. Define

$$a : K^{(\mathbb{N})} \longrightarrow K^{(\mathbb{N})} \qquad b : K^{(\mathbb{N})} \longrightarrow K^{(\mathbb{N})}$$

$$e_n \longmapsto e_{\sigma_0(n)} \qquad e_n \longmapsto e_{\sigma_1(n)}$$

$$x : K^{(\mathbb{N})} \longrightarrow K^{(\mathbb{N})} \quad x(e_n) = \begin{cases} e_{\sigma_0^{-1}(n)} & \text{if } n \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}$$

$y: K^{(\mathbb{N})} \longrightarrow K^{(\mathbb{N})}$ $\quad y(e_n) = \begin{cases} e_{\sigma_1^{-1}(n)} & \text{if } n \in \mathbb{N}_1 \\ 0 & \text{otherwise.} \end{cases}$

We will show that $a \circ x + b \circ y = id$. ETPT $a \circ x + b \circ y$ operating on $e_n$ is $e_n$ $\forall\, n \in \mathbb{N}$. Let $n \in \mathbb{N}_0$, $x(e_n) = e_{\sigma_0^{-1}(n)}$

$a(x(e_n)) = e_{\sigma_0(\sigma_0^{-1}(n))} = e_n$.

$y(e_n) = 0 \Rightarrow b(y(e_n)) = 0$. Therefore, $\cancel{a_0}$

$(a \circ x + b \circ y)(e_n) = e_n \quad \forall\, n \in \mathbb{N}_0$.

Let $n = \mathbb{N}_1$. $\quad x(e_n) = 0 \Longrightarrow a(x(e_n)) = 0$

$y(e_n) = e_{\sigma_1^{-1}(n)}$, $\quad b(y(e_n)) = e_n$.

Therefore, $\quad a \circ x + b \circ y = id$.

Suppose $A$ is a division ring, then any module $V$ over $A$ has a rank.

Let $A$ be a finite ring and let $V$ be a finitely generated free $A$-module. Then, $V$ has a rank.

Proof: Let $x_1, x_2, \ldots x_n \in V$, be the basis of $V$ over $A$. We construct a bijection from

$$A^n \longrightarrow V.$$

$$(a_1, a_2 \ldots a_n) \longmapsto a_1 x_1 + a_2 x_2 \ldots a_n x_n.$$

The above map is surjective because $\{x_1, x_2 \ldots x_n\}$ is a generating set. Also, the map is injective due to linear independence of $(x_1, x_2, \ldots x_n)$.

Therefore, $\quad |V| = |A^n| = |A|^n$.

Since $A$ is finite, any other basis must have the same cardinality $(n = m)$. Therefore, $V$ has a rank.