GENERATING SYSTEM FOR MODULES

A is a ring, V is a A-module, $\underline{x} = \{x_i \mid i \in I\} \subseteq V$.
The A-submodule of V generated by $\underline{x}$ is denoted

$$\sum_{i \in I} A x_i = \left\{ \sum a_i x_i \;\middle|\; (a_i) \in A^{(I)} \right\}$$

V is generated by $\underline{x}$ or $\underline{x}$ is the generating system of V if $V = \sum_{i \in I} A x_i$

V is finitely generated if $\exists$ a finite generating system $y_1, y_2, \ldots y_n$ such that $V = Ay_1 + Ay_2 \ldots Ay_n$.

Example: $A = \mathbb{Z}$, $V = \mathbb{Z}$, The sets $\{1\}$ and $\{2,3\}$ are generating systems. A minimal generating system is one in which no proper subset can be a generating system. The sets $\{1\}$ and $\{2,3\}$ are minimal generating systems of $\mathbb{Z}$.

$$\mu_A(V) = \min\{\,|\underline{x}| \mid \underline{x} \text{ is a generating system for } V\}$$

$\rightarrow$ the minimal number of generators for V.

Example (i) $\mu_{\mathbb{Z}}(\mathbb{Z}) = 1$

(ii) $A = \mathbb{Z}$, $V = \mathbb{Q}$ $\mu_A(\mathbb{Z})$ does not exist.

$\mathbb{Q}$ has no minimal no: of generators. Moreover, if $x_i, i \in I$ generates $\mathbb{Q}$ & $J \subseteq I$ such that $I \setminus J$ is finite, then $x_j, j \in J$ still generates $\mathbb{Q}$. The proof is based on the fact that the quotient group of any divisible group is divisible.

Let $Q = \sum_{i \in I} \mathbb{Z} x_i$, $H = \sum_{i \in J} \mathbb{Z} x_j$ To show $Q \cong H$.

H is abelian and divisible, i.e, $\forall\, a \in H$ and $\forall\, n \in \mathbb{N}$, $\exists$ a $b \in H$ such that ~~b=~~ $a = n \cdot b$

$$\begin{array}{ccc} \cancel{H} & \xrightarrow{\quad} & H \\ \downarrow & & \cancel{\downarrow} \\ H/K & & \end{array}$$

Lemma: Suppose V is finitely generated ie, $V = Ay_1 + Ay_2 \ldots Ay_n$. Then there exists a finite generating system within every generating system $\underline{x} = \{x_i \mid i \in I\}$.

Proof:

$$y_j = \sum_{i \in E(j)} a_{ij} x_i \quad \{x_i \mid i \in I\} \text{ is a generating system}$$

Note $E(j)$ is finite. ~~beca~~ $\Longrightarrow E = \bigcup_{j=1}^{n} E_j$ is finite.

$$y_1, y_2, \ldots y_n \in \sum_{i \in E} A x_i$$

$$\Longrightarrow \cancel{A}V = Ay_1 + Ay_2 \ldots Ay_n \subseteq \sum_{i \in E} A x_i$$

Therefore, $V = \sum_{i \in E} A x_i$

22.1 Theorem: Suppose that $\underline{y}$ is an infinite generating system for V. Then every generating system $x_i, i \in I$ contains a generating system $x_j, j \in J$ with $|J| \leq |\underline{y}|$

Proof: Let $y \in \underline{y}$. Then $\exists$ a finite subset $E(y)$ of I such that

$$y = \sum_{i \in E(y)} a_i x_i$$

Let $J = \bigcup_{y \in \underline{y}} E(y)$. $\quad y \in \sum_{j \in J} \overset{A}{\cancel{A_j}} x_j \quad \forall y \in \underline{y}$

$$\Longrightarrow V = \sum_{j \in J} A x_j$$

In order to show that $|J| \leq |\underline{y}|$, we use the following corollary.

$|N_i| \leq M$

Corollary 2 M is infinite set, N_i, $i \in I$ is a family of ~~finite~~ sets, $|I| \leq |M|$. Then $\left|\bigcup_{i\in I} N_i\right| \leq |M|$

Proof: We must find a surjective map

$$g : M \longrightarrow \bigcup_{i \in I} N_i$$

Assume N_i are non-empty. Since M is infinite, there exists a surjective map

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend right, "g_i"'] & \cancel{N} \arrow[r] & N_i \end{tikzcd}$$

g_i — Surjective map.

Therefore, the map
$$M \times I \longrightarrow \bigcup_{i\in I} N_i$$
$$(x, i) \longrightarrow g_i(x) \quad \text{is surjective.}$$

$\Longrightarrow \quad |M \times I| \geq \left|\bigcup_{i\in I} N_i\right|$. However, since M is infinite, and $|I| \leq |M|$, $|M \times I| = |M|$. This can be shown by the following corollary.

$$\Longrightarrow \left|\bigcup_{i\in I} N_i\right| \leq |M|$$

Corollary 1 M, N are non-empty sets, one of which is infinite, then $|M \times N| = \sup\{|M|, |N|\}$.

Proof: We may assume $|M| \leq |N|$, and N is infinite. Since $|M| \leq |N|$, there exists a injective map $M \overset{f}{\hookrightarrow} N$.

$$N \xrightarrow{\text{injective}} M \times N \xrightarrow{f \times id} N \times N$$

$n \longmapsto (m_0, n)$ $\quad f \times id$ is injective because f is injective.

$\Longrightarrow \quad |N| \leq |M \times N| \leq |N \times N| = |N| \quad \Longrightarrow \quad |M \times N| = |N|$

The result: $|N \times N| = |N|$ comes from the following theorem.

Theorem: M is an infinite set. Then $|M \times M| = |M|$.

Corollary 3 N is a infinite set and $f: M \to N$ is a map with finite fibres ie $y \in N$, $f^{-1}(y)$ is finite where $f^{-1}(y)$ is called ~~finite~~ fibre over y.
Then $|M| \leq |N|$.

Proof:

$f^{-1}(y)$ is finite. $\quad M = \bigcup_{y \in N} f^{-1}(y)$

Using corollary 1, we get $|M| = \left|\bigcup_{y \in N} f^{-1}(y)\right| \leq |N| \Rightarrow |M| \leq |N|$

Corollary 4: M is an infinite set. Then $|M^n| = |M|$ $\forall\, n \in \mathbb{N}$.

$$|M^n| = |\underbrace{M \times M \times \ldots \ldots M}_{n \text{ times}}| = |M|.$$

Proof: Show this by induction.

Corollary 5: M is infinite $\mathcal{P}_f(M) = \{N \subseteq M \mid |N| \text{ is finite}\}$

The $|\mathcal{P}_f(M)| = |M|$.

proof: $n \in \mathbb{N}^*$ $\quad \mathcal{P}_n(M) = \{N \subseteq M \mid |N| \leq n\}$

$\mathcal{P}_f(M) = \{\phi \cup \bigcup_{n \in \mathbb{N}^*} \mathcal{P}_n(M)\}$. ETPT $|\mathcal{P}_f(M)| \leq |M|$

$\forall$ $|M| \leq |\mathcal{P}_1(M)| \leq |\mathcal{P}_n(M)|$

Consider the map $M^n \longrightarrow \mathcal{P}_n(M)$

$(x_1, x_2 \ldots x_n) \longmapsto \{x_1, x_2, \ldots x_n\}$

Surjective

$\Longrightarrow |\mathcal{P}_n(M)| \leq |M^n| = |M|$

$\Rightarrow |\mathcal{P}_n(M)| \leq |M|$ $\forall n \in \mathbb{N}^*$.

Therefore $|\mathcal{P}_f(M)| = \left|\bigcup_{n \in \mathbb{N}} \mathcal{P}_n(M)\right| = |M|$ by using the corollary 2.