DIMENSION AND RANK

23.1 **Theorem**: Let M be an infinite set. Then $|M \times M| = |M|$. i.e, there is a bijection between $M \longrightarrow M \times M$.

proof:
$$\mathfrak{M} = \{ (X, f) \mid X \subseteq M,\ f : X \xrightarrow{\cong} X \times X \text{ bijective} \}$$
$\mathfrak{M} \neq \emptyset$ because $\{ (\{x\}, i_x) \in \mathfrak{M} \}$ $\quad x \in M$
$$i_x : \{x\} \to \{(x, x)\}$$
We define an order on $\mathfrak{M}$.
$(X, f), (Y, g) \in \mathfrak{M}$. Define $(X, f) \le (Y, g) \iff$
$X \subseteq Y$, and $f = g/X$.
Verify that the above definition makes $\mathfrak{M}$ an ordered set. The set $\mathfrak{M}$ has a maximal element (To be proved}. by Zorn's lemma).

Zorn's lemma: Let $(X, \le)$ be an ordered set, and inductively ordered, then ~~it~~ X has a maximal element.

Inductively ordered $\Rightarrow$ If every chain in X has an upper bound in X. ~~Every~~ ch
Every chain in X is a totally ordered ~~set~~ subset of X.
Note $(\mathbb{N}, \le)$ is not inductively ordered).
Maximal element: No element is bigger than the maximal element. There may be many maximal elements.
ETPT: $\mathfrak{M}$ is inductively ordered and ~~$\Rightarrow$~~ a maximal element is (M, f).
Let $\mathcal{C}$ be a chain in $(\mathfrak{M}, \le)$. $\mathcal{C} = \{ (X_i, f_i) \mid i \in I \}$
Look for $(X, f) \in \mathfrak{M}$ such that $(X_i, f_i) \le (X, f)\ \forall i \in I$
$\Rightarrow X_i \subseteq X, \quad f_i = f/X_i \quad \forall i \in I$
Let $X = \bigcup_{i \in I} X_i$. Define $f : X \longrightarrow X \times X \subseteq M \times M$
$$x \longmapsto f_i(x)$$
Verify that f is well defined. Note that if $x \in X_i \cap X_j$

Note that if $x \in X_i \cap X_j \Rightarrow (X_i, f), (X_j, f)$ are comparable (because they are elements in a chain).
if $(X_i, f_i) \leq (X_j, f_j) \Rightarrow$ ~~f_i(~~ $f_i = f_j|X_i$

Therefore, (X, f) is an upper bound of the chain.
$\Rightarrow \mathfrak{M}$ is inductively ordered. By Zorn's lemma, $\exists$ a maximal element ~~of~~ (Z, h) in the chain.

We will show that $|Z| = |M|$. Suppose $Z \subsetneq M$, we produce an $(Z_1, h_1) \gneq (Z, h)$, thereby contradicting the maximality of (Z, h). $Z \subsetneq M \Rightarrow M \setminus Z \neq \phi$
Claim: $|M \setminus Z| \leq |Z|$

Proof: Assume the contrary, $|Z| \leq |M \setminus Z|$, ~~i.e, $\exists$ a~~
i.e, $\exists$ a $Z_1 \subseteq M \setminus Z$ such that $Z \xrightarrow[\text{bijection}]{\cong} Z_1 \subseteq M \setminus Z$

$|Z| = |Z_1|$. Consider a mapping from

$$(Z \cup Z_1) \xrightarrow{\cong} (Z \cup Z_1) \times (Z \cup Z_1)$$
$$= (Z \times Z) \uplus (Z \times Z_1) \uplus (Z_1 \times Z) \uplus (Z_1 \times Z_1)$$

There already exists a bijection $Z \xrightarrow{h} Z \times Z$.
There also exists a bijection from

$$Z_1 \longrightarrow (Z \times Z_1) \uplus (Z_1 \times Z) \uplus (Z_1 \times Z_1)$$

Note $|Z| = |Z_1|$. Therefore $|Z \times Z_1| = |Z_1 \times Z| = |Z_1 \times Z| = |Z \times Z|$.

It is clear that $|Z| = |Z \times Z| = |Z \times Z_1| = |Z_1 \times Z| = |Z_1 \times Z|$.
By the lemma following the proof.

$$|Z_1| = |(Z \times Z_1) \uplus (Z_1 \times Z) \uplus (Z_1 \times Z_1)|$$

Therefore, $\exists$ a bijection g

$$Z_1 \xrightarrow{g} (Z_1 \times Z) \uplus (Z \times Z_1) \uplus (Z_1 \times Z_1)$$

Combining the bijections h and g, a bijection h_1 from

$$(Z \cup Z_1) \xrightarrow{h_1} (Z \cup Z_1) \times (Z \cup Z_1)$$

It is obvious that $(Z, h) \leqslant (Z \cup Z_1, h_1)$ contradicting the maximality of (Z, h). Therefore, $|M \setminus Z| \leqslant |Z|$

$$M = M \setminus Z \cup Z \Rightarrow |M| = |M \setminus Z \cup Z|$$

Since $|M \setminus Z| \leqslant |Z|$, by the ~~following~~ lemma following this proof $|M| \leqslant |Z|$. Since $Z \subsetneq M$, it is clear that $|Z| \leqslant |M|$. Therefore $|M| = |Z|$.

There exists a bijection between M and Z.

$$\exists\ Z \xrightarrow{h} Z \times Z \Rightarrow |Z| = |Z \times Z|$$

$$|Z| = |M| \Rightarrow Z \xrightarrow{\simeq} M \Rightarrow |Z \times Z| = |M \times M|$$

$$\Rightarrow |M| = |M \times M|.$$

Lemma: If Z is a set such that $|Z| = |Z \times Z|$, Q_j, $j \in J$ be a family of sets J countable, ~~then~~ ~~if $|Q_j| \leqslant |Z|$, $\left|\bigcup_{j \in J} Q_j\right| \leqslant |Z|$.~~ $|Q_j| \leqslant |Z|$, then $\left|\bigcup_{j \in J} Q_j\right| \leqslant |Z|$

<u>Proof</u>: Since $|Q_j| \leqslant |Z|$, there exists a injective mapping $Q_j \xrightarrow{f_j} Z \quad \forall j$

· Since J is countable, $|J| \leqslant |\mathbb{N}|$ $\quad \mathbb{N}$ - natural numbers

Let $Q = \bigcup_{j \in J} Q_j$. The map $Q \xrightarrow{f} Z \times J$

$$x \longmapsto (f_j(x), \cancel{x}) \quad (f_j(x), j)$$

injective.

Therefore $|Q| \leqslant |Z \times J| \leqslant |Z \times \mathbb{N}| \leqslant |Z \times Z| = |Z|$.