Degree of a Polynomial

Let A be a commutative ring, and I be any set. The monoid ring of $\mathbb{N}^{(I)}$ over A denoted by $A[\mathbb{N}^{(I)}]$. This is also a free module with basis $e_\nu,\ \nu \in \mathbb{N}^{(I)}$ $\quad e_\nu(\tau) = \begin{cases} 1 & \text{if } \nu = \tau \\ 0 & \text{otherwise} \end{cases}$

or $e_\nu = (\delta_{\nu\tau})_{\tau \in \mathbb{N}^{(I)}}$

~~$\varepsilon_i(j)$~~ Let ε_i ~~be~~ denote the i^{th} basis of $\mathbb{N}^{(I)}$.

$\varepsilon_i(j) = \delta_{ij}$. $\quad e_\nu : \mathbb{N}^{(I)} \longrightarrow A$

$\varepsilon_i \longmapsto e_{\varepsilon_i} = X_i$

$$\nu \in \mathbb{N}^{(I)} \Rightarrow \nu = \sum_{\substack{\cancel{i \in \mathbb{N}^{(I)}} \\ i \in I}} \nu_i \varepsilon_i \Rightarrow e_\nu = \prod_{i \in I} X_i^{\nu_i} = X^\nu.$$

Any element f of $A[\mathbb{N}^{(I)}]$ can be written as

$$f = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu e_\nu = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X^\nu, \text{ where } X^\nu = \prod_{i \in I} X_i^{\nu_i}$$

$A[\mathbb{N}^{(I)}]$ is also called the polynomial ring over A in $X_i,\ i \in I$ and is denoted by $A[X_i \mid i \in I]$. Note that every polynomial f is composed of finite sum of monomials. Verify that if $f = \sum_\nu a_\nu X^\nu$, $g = \sum_\nu b_\nu X^\nu$

(i) $f+g = \sum_\nu (a_\nu + b_\nu) X^\nu$, (ii) $f \cdot g = \sum_\nu c_\nu X^\nu$, where

$$c_\nu = \sum_{\nu = \lambda + \mu} a_\lambda b_\mu$$

Two polynomials ~~a.~~ f and g are equal $\Leftrightarrow a_\nu = b_\nu\ \forall \nu \in \mathbb{N}^{(I)}$

Universal property of $A[X_i \mid i \in I]$.

Given an A-algebra B, and a family of elements $\{x_i \in B \mid i \in I\}$ denoted by $\underline{x}$, $\exists$ a unique isomorphism

$$A[X_i \mid i \in I] \xrightarrow{\phi_{\underline{x}}} B$$
$$X_i \longrightarrow x_i$$

By this map $f \longrightarrow \sum_{\nu} a_\nu \prod_{i \in I} x_i^{\nu_i} = f(\underline{x})$

Verify that (i) $\phi_{\underline{x}}(1) = 1$, (ii) $\phi_{\underline{x}}(f+g) = \phi_{\underline{x}}(f) + \phi_{\underline{x}}(g)$,
(iii) $\phi_{\underline{x}}(f \cdot g) = \phi_{\underline{x}}(f) \cdot \phi_{\underline{x}}(g)$, (iv) $\phi_{\underline{x}}(0) = 0$.

$f(\underline{x})$ is called the value of f at $\underline{x}$, and is an element in B. $\underline{x}$ is called a zero of f if $f(\underline{x}) = 0$.
Remark: $A = \mathbb{Q}$, for a polynomial $f \in \mathbb{Q}[X_1, X_2]$ its zero may exist in $\mathbb{Q} \times \mathbb{Q}$.

Let $\nu = \mathbb{N}^{(I)}$, $|\nu| = \sum_{i \in I} \nu_i$, $X^\nu = \prod_{i \in I} X_i^{\nu_i}$,

a homogenous polynomial of degree n is a $f \in A[\mathbb{N}^{(I)}]$ such that $f = \sum_{|\nu| = n} a_\nu X^\nu$ where $(a_\nu) \in A^{(\mathbb{N}^{(I)})}$

Example: $f = \underbrace{2X + 3Y}_{\text{degree } 1} + \underbrace{5X^2 + 7XY}_{\text{degree } 2} + \underbrace{5Y^3}_{\text{degree } 3}$

Therefore, any f can be broken down into
$f = f_0 + f_1 + f_2 \ldots f_m$, where f_m = form of degree n in f.

Degree of X_i in f: $A[\mathbb{N}^I] = A[X_j \mid j \in I \setminus \{i\}][X_i]$

$\Rightarrow$ Any element ~~of~~ $f = g_0 + g_1 X_i + g_2 X_i^2 \ldots g_m X_i^m$,
where $g_0, g_1, g_2 \ldots g_m \in A[X_j \mid j \in I \setminus \{i\}]$.
The degree of X_i in f is m. or $\deg_{X_i} f = m$.

Leading coefficient: If $f = a_0 + a_1 X + a_2 X^2 \ldots a_d X^d$
$\Rightarrow a_d$ is the leading coefficient. It is easy to show that (a) $\deg(f+g) \leq \max(\deg f, \deg g)$, ~~(b) $\deg(fg)$~~
(b) $\deg(fg) \leq \deg f + \deg g$.

~~A~~ Example: $A = \mathbb{Z}_4$, $f = 2X$, $g = 2X$ $f \cdot g = 0$

$$\deg(f \cdot g) = \deg(0) \leq \deg f + \deg(g)$$

(By choice?) $\deg(0) = -\infty$

$$\Rightarrow -\infty \leq 2$$

23.1 Proposition: A is an integral domain, $f, g \in A[X_i \mid i \in I] = P$. Then $\deg(f \cdot g) = \deg f + \deg g$
In particular, P is an integral domain.

Proof: WMA $I = \{1\}$ $X_1 = X$.

$f = a_d X^d + \ldots a_0,\ a_d \neq 0$

$g = b_{d'} X^{d'} + \ldots b_0,\ b_{d'} \neq 0$ $\Rightarrow$ ~~ad~~ $a_d b_{d'} \neq 0$ because A is integral domain

$\Rightarrow \deg f \cdot g = \deg f + \deg g$. because the maximum power of X comes from $a_d b_{d'} X^{d+d'}$.

P is never a field. If P is a field $\Rightarrow X_i f = 1$

but $\underbrace{\deg(X_i) + \deg(f)}_{\geq 1} = 0$, i.e $0 \geq 1$

Hence, by this contradiction P is never a field.

$P^\times = A^\times$ if A is an integral domain.

$$f \cdot g = 1 \Rightarrow \deg(f) + \deg(g) = 0 \Rightarrow \deg(f) = \deg(g) = 0$$

$$\Rightarrow f \in A^\times, \text{ and } g \in A^\times.$$

If A is not an integral domain, there may be non-constant polynomials which have inverses.

Example: $A = \mathbb{Z}_4$, then $(1 + 2X) \cdot (1 - 2X) = 1$

$$(1 + 2X^2)(1 - 2X^2) = 1$$

K is a field. $K[X_i \mid i \in I]$ is an integral domain but not a field. Since $K[X_i \mid i \in I]$ is an I.D, there exists a quotient field denoted by $K(X_i \mid i \in I)$. The elements of this field are of the form f/g, $g \neq 0$, and they are called rational functions.