Bases for Vector spaces.

Let K be a division ring and V a K-vector space.

Lemma 1 Let $\{x_i \mid i \in I\}$ be a linearly independent system in V and $x \in V$. Then

$$\{x_i \mid i \in I\} \cup \{x\} \text{ is linearly dependent} \iff x \in \sum_{i \in I} K x_i$$

Proof: Use the fact that K is a divison ring.

Lemma 2: Let $\{x_i \mid i \in I\} \subseteq V$. The following are equivalent (TFAE):

(i) $\{x_i \mid i \in I\}$ is a basis of V

(ii) $\{x_i \mid i \in I\}$ is a maximal linearly independent set

(iii) $\{x_i \mid i \in I\}$ is a minimal generating system.

Proof: (i) $\iff$ (ii) from Lemma 1

(i) $\implies$ (iii) from Lemma 1

To prove (iii) $\implies$ (i). We will show that ~~t~~ a minimal generating system is linearly independent (LI):

Consider $\sum_{i \in I} a_i x_i = 0$. If linearly dependent,

$(a_i) \in A^{(I)}$, $(a_i) \neq 0$. Let $a_m \neq 0$, $m \in I$. Then

$$x_m = a_m^{-1} \sum_{\substack{i \in I \\ i \neq m}} (-a_i) x_i$$

$\implies x_m \in \sum_{\substack{i \in I \\ i \neq m}} K x_i$. This contradicts the fact that $\{x_i \mid i \in I\}$ is a minimal generating system.

Therefore, $\{x_i \mid i \in I\}$ is linearly independent $\Rightarrow$ $\{x_i \mid i \in I\}$ is a basis.

25.1 Theorem: Suppose V is finitely generated. Then V is a free-module, i.e. V has a basis. Moreover,

1. Every generating system contains a basis.
2. Every basis of V has finitely many elements.
3. Every linearly independent system of elements in V can be extended, using elements from an arbitrary given generating system of V, to a basis of V.

(In particular, V has a finite basis).

Proof: Let $\{x_i \mid i \in I\}$ be a generating system for V. Then $\exists$ a ~~subset~~ finite subset $J \subseteq I$, such that $\{x_j \mid j \in J\}$ generates V. Choose $J' \subseteq J$ such that $\{x_j \mid j \in J'\}$ is a minimal generating system for V. This proves (1).

To prove (2) use ~~the~~ (1) and Lemma 1.

To prove (3):
$\{x_i \mid i \in I\}$ linearly independent system and $\{y_j \mid j \in J\}$ (J finite) is an arbitrary ~~sy~~ generating system for V. Then $\exists$ a $J' \subseteq J$ such that if $\{x_i \mid i \in I\} \cup \{y_j \mid j \in J'\}$ is a linearly independent system. Therefore,

$$y_j \in \sum_{i \in I} K x_i + \sum_{j \in J'} K y_j \quad \forall j \in J$$

Therefore,

$$\sum_{i \in I} K x_i + \sum_{j \in J'} K y_j = V$$

$\Rightarrow$ $\{x_i \mid i \in I\} \cup \{y_j \mid j \in J\}$ is a basis.

<u>Illustrative Example</u>: Let $K = \mathbb{Z}$ (not a division ring). The set $\{2, 3\}$ is a minimal generating system but it is not linearly independent.

Remark: V is ~~deno~~ called finite if V is finitely generated. Note $A[X]$ is A-module and is infinitely generated, but as an algebra it is finitely generated.

Corollary 1: V is finite K-module, and $W \subseteq V$ is a K-submodule. Then W is also finite. In fact, every basis of W is contained in a basis of V.

<u>Proof</u>: If W is not finitely generated, then $\exists$ a $(y_n)_{n \in \mathbb{N}} \in W$ which is linearly independent. This set cannot be extended to a basis of V, which has only finite number of elements. Therefore, by this contradiction <u>W is also finite</u>.

Using ~~pa~~ (3) of Theorem 25.1, the second part of the corollary can be proved.

<u>25.2 Theorem</u>: Let V be any K-vector space (need not be finitely generated). Let $\{x_i \mid i \in I\}$ be a generating system of V such that $\{x_j \mid j \in J\}$ $(J \subseteq I)$ is linearly independent. Then $\exists$ a N such that $J \subseteq N \subseteq I$ $(x_i)_{i \in N}$ is a basis of V.

<u>Proof</u>: Let $\mathfrak{X} = \{X \subseteq I \mid J \subseteq X$, and $(x_i)_{i \in X}$ are linearly independent$\}$.

$J \in \mathfrak{X} \neq \phi$. $\mathfrak{X}$ is ordered by natural inclusion. Let $\mathfrak{K}$ be a non-empty chain in $\mathfrak{X}$.

$$\mathfrak{K} = \{K \subset I \mid (x_k)_{k \in K} \text{ linearly independent}\}$$

Note that $J \subseteq K$. It is clear that $L = \bigcup_{K \in \mathcal{R}} K$ is an upper bound for $\mathcal{R}$ in $\mathfrak{X}$. We must show that the elements in L are linearly independent.

union of all sets

Consider $a_{i_1} x_{i_1} + a_{i_2} x_{i_2} \cdots\cdots a_{i_n} x_{i_n} = 0$

Note that since the sets in the chain are completely ordered by natural ~~in~~ inclusion, ~~and~~ there must be one set in the chain which contains all the elements. Therefore, $\exists K \in \mathcal{R}$ $x_{i_1}, x_{i_2}, \ldots x_{i_n} \in K$.

$\Longrightarrow a_{i_1} = a_{i_2} = \cdots\cdot a_{i_n} = 0$.

By Zorn's lemma, $\exists$ a maximal element $N \in \mathfrak{X}$. Obviously, $N \supseteq J$, and $(x_i)_{i \in N}$ is linearly independent. By part (2) of Lemma 2, $(x_i)_{i \in N}$ is a basis.

Illustrative Example: K is a field, and $k \subseteq K$ is a subfield. Then K is a k-vector space. Infact, K is a k-algebra, $\exists$ a k-basis for K. Note $1_k = 1_K$. For instance, $\exists$ a $\mathbb{Q}$-basis for $\mathbb{R}$ which is called Hammel basis.

Corollary: V any K-vector space, and $W \subseteq V$ be any K-submodule. Then

1) Every generating system of V contains a basis of V.
2) V has a basis.
3) Every basis of W can be extended to a basis of V.
4) There exists a basis of W which can be extended to a basis of V.