Dimension Of Vector Spaces

27.1 Theorem: A is a ring and V is a A-module. Suppose $\underline{y}$ is an infinite generating system for V. Then every generating system $\{x_i \mid i \in I\}$ contains a generating system $\{x_j \mid j \in J,\ J \subseteq I\}$ with $|J| \leqslant |\underline{y}|$.

Proof: Proved earlier in class 22 (Theorem 22.1)

We have already shown that for a division ring K, a K-vector space V has a basis.

Dimension of V: All K-bases of V have the same cardinality. This cardinality is called the dimension of V, and is denoted by $\mathrm{Dim}_K V$.

Proof: Case 1: V is infinite.

Let $\{x_i \mid i \in I\} = \underline{x}$ and $\underline{y} = \{y_j \mid j \in J|$ be two basis of V and I is infinite. To prove that (TPT) $|I| = |J|$. From the above theorem $|J| \leqslant |I|$. Interchange the roles of I and J, and we get $|I| \leqslant |J|$ $\Longrightarrow |I| = |J|$.

Case 2: V is finite. Let $\{x_1, x_2, \ldots x_n\}$, $\{y_1, y_2 \ldots y_n\}$ be two basis, ETPT $m \leqslant n$. (Interchange $\underline{x}$ and $\underline{y}$).

27.2 Theorem (Steinitz's ~~th~~ Exchange Theorem).

Let V be a K-vector space, and $\{x_1, x_2, \ldots x_n\}$ be a basis of V and $\{y_1, y_2 \ldots y_m\}$ be a linearly independent set. Then $m \leqslant n$ and $\exists$ $n-m$ elements from $\{x_1, x_2 \ldots x_n\}$ such that $\{y_1, y_2 \ldots y_m\}$ together with these $(n-m)$ elements form a basis of V.

Proof of Steinitz's theorem: Proof by induction on m.

$m = 0$: Nothing to prove. Assume the hypothesis to be true upto $m-1$. Then $m-1 \leq n$ and there are $n-m+1$ elements from $\{x_1, x_2, \dots x_n\}$ so that (without loss of generality)

$$\{y_1, y_2, \dots y_{m-1}, x_m, x_{m+1}, \dots x_n\} \text{ is a basis of } V.$$

Now consider two cases. Case 1: $m-1 = n$

This would imply that y_m is linearly dependent on $\{y_1, y_2 \dots y_{m-1}\}$ which is a contradiction. $\Rightarrow m-1 \neq n$

Case 2: $m-1 < n \implies m < n+1 \implies m \leq n$.

This is sufficient to prove that the cardinality of two bases of a finite dimensional vector space is the same.

Since $\{y_1, y_2 \dots y_{m-1}, x_m, x_{m+1}, \dots x_n\}$ forms a basis, $y_m = a_1 y_1 + a_2 y_2 \dots a_{m-1} y_{m-1} + b_m x_m \dots b_n x_n$

Without loss of generality, let $b_m \neq 0$. Since K is a division ring

$$x_m = b_m^{-1}(y_m - a_1 y_1 - a_2 y_2 \dots - b_{m+1} x_{m+1} \dots - b_n x_n)$$

$$\implies x_m \in \{Ky_1 + Ky_2 \dots Ky_{m-2} + Ky_{m-1} + Ky_m + Kx_{m+1} \dots Kx_n\}$$

Since $\{y_1, y_2, y_{m-2}, y_{m-1}, y_m, x_m, x_{m+1}, \dots x_n\}$ belong to the above sub-module,

$$Ky_1 + Ky_2 \dots Ky_{m-1} + Ky_m + Kx_{m+1}, \dots Kx_n = V$$

To prove that they are linearly independent.
Consider

$$0 = a_1 y_1 + a_2 y_2 \dots a_m y_m + b_{m+1} x_{m+1} \dots b_n x_n$$

If all a_i are 0, then all b_j are 0 (x_i are LI)
Similarly if all $b_i = 0$, then $a_i = 0$ (y_i are LI)
Assume $a_m \neq 0$, then $y_m \in Ky_1 + \dots Ky_{m-1} + Kx_{m+1} \dots Kx_n$

$\Rightarrow Ky_1 + \dots Ky_{m-1} + \dots Kx_{m+1} \dots Kx_n = V$. ~~But~~ {

This contradicts the hypothesis $\{y_1, y_2 \dots y_{m-1}, x_m, \dots x_n\}$ is a basis and therefore a minimal generating system.
Therefore, $a_m = 0$.

Corollary 1: Let V be a K-vector space, $\mathrm{Dim}_K V = n$, and $\{x_1, x_2 \dots x_n\}$ be elements in V. The following are equivalent (a) $\{x_1, x_2, \dots x_n\}$ is a basis, (b) $\{x_1, x_2 \dots x_n\}$ is a generating system, ~~and~~ (c) $\{x_1, x_2 \dots x_n\}$ is linearly independent.

Proof: (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c). Use the Exchange theorem to show that (c) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c)

Corollary 2: Let V be a K-vector space. Suppose $\{x_1, x_2 \dots x_n\} \subseteq V$, and $W = Kx_1 + Kx_2 \dots Kx_n$.
Then $\mathrm{Dim}_K W =$ maximum no. of elements in $\{x_1, \dots x_n\}$ which are linearly independent

Corollary 3: Let V be a ~~infinite~~ finite dimensional K-vector space. W is a K-subspace. Then $\mathrm{Dim}_K W \leq \mathrm{Dim}_K V$.
Moreover, it is an inequality iff $W = V$.

Proof: Let $\{x_1, x_2 \dots x_m\} \in W$ be a basis of W. It can be extended to $\{x_1, x_2 \dots x_m, x_{m+1} \dots x_n\}$ as a basis of V. Therefore

$$\mathrm{Dim}_K W = m \leqslant n = \mathrm{Dim}_K V.$$

If it is equal, then no extension is necessary.

If V is a K-vector space, & U and W are K-subspaces of V, U and W are finite, and $U \subseteq W$.

Then $\quad U = W \iff \mathrm{Dim}_K U = \mathrm{Dim}_K W.$

Illustrative Example: Let V be a countable infinite dimensional K-vector space, i.e V has a countable basis. Let $\{x_n \mid n \in \mathbb{N}\}$ be a basis of V.

The K-subspace generated by $\{x_{2n} \mid n \in \mathbb{N}\}$, i.e,

$$W = \sum_{n \in \mathbb{N}} K x_{2n}.$$

It is clear that $W \subsetneq V$, but $\mathrm{Dim}_K W = |\mathbb{N}| = \mathrm{Dim}_K V$.

Example: K is a division ring, and I any set.

$K^{(I)} \subseteq K^I$. For $K^{(I)}$ $e_i, i \in I$ is ~~a~~ the standard basis. Every $a \in K^{(I)}$ can be expressed as

$$a = (a_i) = \sum_{i \in I} a_i e_i$$

(Note the summation is only for finite terms)

$\mathrm{Dim}_K K^{(I)} = |I|$. In particular if $I = \{1, 2 \dots n\}$

$$\mathrm{Dim}_K K^n = |I| = n.$$

However, it is not trivial for K^I. We have already shown that a basis exists. Example for K^I could be $C_{\mathbb{R}}([0,1])$: set of continuous functions on $[0,1]$.