

Algebra, Arithmetic and Geometry – With a View Toward Applications / 2005
Lectures : Tuesday/Thursday 18:15–19:15 ; LH-1, Department of Mathematics

0.A Sets and Maps



Georg Ferdinand Ludwig Philipp Cantor[†]
(1845-1918)

0.A.1. 1). For a set A the following statements are equivalent: (i) $A = \emptyset$. (ii) $A - B = A \cap B$ for every set B . (iii) There exists a set B with $A \setminus B = A \cap B$. (iv) $B \setminus A = B \cup A$ for every set B . (vi) There exists a set B with $B \setminus A = B \cup A$.

2). For sets A and B the following statements are equivalent: (i) $A \subseteq B$. (ii) $A \cap B = A$. (iii) $A \cup B = B$. (iv) $A \setminus B = \emptyset$. (v) $B \setminus (B \setminus A) = A$. (vi) $A \cup (B \cap C) = (A \cup C) \cap B$ for every set C . (vii) There exists a set C with $A \cup (B \cap C) = (A \cup C) \cap B$.

3). For sets A, B show that $(A \cap B) \cap (A \setminus B) = \emptyset$ and $(A \cap B) \cup (A \setminus B) = A$.

4). For sets A, B, C show that:

a). $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$. **b).** $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$. **c).** $(A \setminus B) \setminus C = A \setminus (B \cup C)$. **d).** $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

5). The sets A, B are equal if and only if $A \cup B = A \cap B$.

6). For sets A, B, C show that:

a). $A \Delta A = \emptyset$, $A \Delta \emptyset = A$. **b).** $A = B$ if and only if $A \Delta B = \emptyset$. $A \cap B = \emptyset$ if and only if $A \Delta B = A \cup B$.

c). $(A \Delta B) \cap (A \cap B) = \emptyset$, $(A \Delta B) \cup (A \cap B) = A \cup B$. **d).** $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$.

e). If $A \Delta B = A \Delta C$, then $B = C$. **f).** $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.

7). The non-empty sets A, B are equal if and only if $A \times B = B \times A$.

8). For sets A, B, C show that:

a). $A \times (B \cup C) = (A \times B) \cup (A \times C)$. **b).** $A \times (B \cap C) = (A \times B) \cap (A \times C)$. **c).** $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

9). Let A, B, C, D be sets. If $A \subseteq C$ and $B \subseteq D$, then show that $A \times B \subseteq C \times D$. Moreover, if $A \neq \emptyset$ and $B \neq \emptyset$, then the converse also holds.

10). For sets A, B, C, D show that: **a).** $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. **b).** $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. Moreover, the equality holds if and only if the conditions “ $A \subseteq C$ or $D \subseteq B$ ” and “ $C \subseteq A$ or $B \subseteq D$ ” are satisfied.

0.A.2. Let $f: X \rightarrow Y$ and let $g: Y \rightarrow X$ be maps. Show that:

a). If gf is injective, then f is injective. **b).** If gf is surjective, then g is surjective. **c).** If gf is bijective, then f is injective and g is surjective. (Give an example to show that neither f nor g is bijective even if gf is bijective.) **d).** If gf is bijective and if f (resp. g) is bijective, then g (resp. f) is also bijective.

0.A.3. a). Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$ and let $h: Z \rightarrow W$ be maps. If gf and hg are bijective, then show that f , g and h are bijective.

b). Let $f: X \rightarrow Y$, $g: Y \rightarrow X$ and let $h: X \rightarrow Y$ be maps. From the equalities $gf = \text{id}_X$ and $hg = \text{id}_Y$, show that equality $f = h$, i.e. the bijectivity of g and $g^{-1} = f = h$.

0.A.4. Let $f: X \rightarrow Y$ be a map.

a). The following statements are equivalent :

(i) f is injective. (ii) For all sets Z and all maps $g_1: Z \rightarrow X$ and $g_2: Z \rightarrow X$, the equality $fg_1 = fg_2$ implies $g_1 = g_2$.

Moreover, if $X \neq \emptyset$, then these statements are further equivalent to: (iii) There exists a map $g: Y \rightarrow X$ such that $gf = \text{id}_X$.

b). The following statements are equivalent:

(i) f is surjective. (ii) For all sets W and all maps $h_1: Y \rightarrow W$ and $h_2: Y \rightarrow W$, the equality $h_1f = h_2f$ implies $h_1 = h_2$. (iii) There exists a map $h: Y \rightarrow X$ such that $fh = \text{id}_Y$. (Such a map h is called a section to f .)

0.A.5. Let $f: X \rightarrow Y$ be a map and let $f_*: \mathfrak{P}(X) \rightarrow \mathfrak{P}(Y)$, $f^*: \mathfrak{P}(Y) \rightarrow \mathfrak{P}(X)$ be the maps induced by f , $A \mapsto f(A)$, $A \subseteq X$, respectively $B \mapsto f^{-1}(B)$, $B \subseteq Y$.

a). The following are equivalent: (i) f is injective. (ii) f_* is injective. (iii) f^* is surjective.

b). The following are equivalent: (i) f is surjective. (ii) f_* is surjective. (iii) f^* is injective.

c). If f bijective, then so are f_* and f^* ; moreover, they are inverses of each other.

0.A.6. Let $f: X \rightarrow Y$ be map and let $X', X'' \subseteq X$, $Y', Y'' \subseteq Y$. Then

a). $f^{-1}(f(X')) \supseteq X'$. b). $f(f^{-1}(Y')) \subseteq Y'$.

c). $f(X' \cup X'') = f(X') \cup f(X'')$, $f(X' \cap X'') \subseteq f(X') \cap f(X'')$ and $f(X' \setminus X'') \supseteq f(X') \setminus f(X'')$.

d). $f^{-1}(Y' \cup Y'') = f^{-1}(Y') \cup f^{-1}(Y'')$, $f^{-1}(Y' \cap Y'') = f^{-1}(Y') \cap f^{-1}(Y'')$ and $f^{-1}(Y' \setminus Y'') = f^{-1}(Y') \setminus f^{-1}(Y'')$.

e). The following are equivalent: (i) f is surjective. (ii) $f(f^{-1}(Y')) = Y'$ for all $Y' \subseteq Y$.

f). The following are equivalent: (i) f is injective. (ii) $f^{-1}(f(X')) = X'$ for all $X' \subseteq X$. (iii) $f(X' \cap X'') = f(X') \cap f(X'')$ for all $X', X'' \subseteq X$. (iv) $f(X') \cap f(X'') = \emptyset$ for all $X', X'' \subseteq X$ with $X' \cap X'' = \emptyset$.

(v) $f(X' \setminus X'') = f(X') \setminus f(X'')$ for all $X', X'' \subseteq X$.

0.A.7. Let $f: X \rightarrow Y$ and let $g: Y \rightarrow Z$ be maps. Then $(gf)(X') = g(f(X'))$ and $(gf)^{-1}(Z') = f^{-1}(g^{-1}(Z'))$ for all $X' \subseteq X$ resp. $Z' \subseteq Z$.

0.A.8. Let X, Y be sets, $Y \neq \emptyset$. There is a surjective map from X onto Y if and only if there is an injective map from Y into X .

0.A.9. Let A_i , $i \in I$, and let B_j , $j \in J$, be families of sets with $I \neq \emptyset \neq J$. Then:

a). $(\bigcap_{i \in I} A_i) \cup (\bigcap_{j \in J} B_j) = \bigcap_{(i,j) \in I \times J} (A_i \cup B_j)$, $(\bigcap_{i \in I} A_i) \cup (\bigcup_{j \in J} B_j) = \bigcap_{i \in I} (\bigcup_{j \in J} A_i \cup B_j)$.

b). $(\bigcup_{i \in I} A_i) \cap (\bigcup_{j \in J} B_j) = \bigcup_{(i,j) \in I \times J} (A_i \cap B_j)$, $(\bigcup_{i \in I} A_i) \cap (\bigcap_{j \in J} B_j) = \bigcup_{i \in I} (\bigcap_{j \in J} A_i \cap B_j)$.

c). $(\bigcup_{i \in I} A_i) \setminus (\bigcup_{j \in J} B_j) = \bigcup_{i \in I} (\bigcap_{j \in J} (A_i \setminus B_j))$, $(\bigcup_{i \in I} A_i) \setminus (\bigcap_{j \in J} B_j) = \bigcup_{(i,j) \in I \times J} (A_i \setminus B_j)$.

Further, if A is another set, then $A \setminus (\bigcup_{j \in J} B_j) = \bigcap_{j \in J} (A \setminus B_j)$ and $A \setminus (\bigcap_{j \in J} B_j) = \bigcup_{j \in J} (A \setminus B_j)$. In particular, the complement of a union is equal to the intersection of the complements and the complement of an intersection is equal to the union of the complements.

0.A.10. Let $f: A \rightarrow B$ be a map and let A_i , $i \in I$, resp. B_j , $j \in J$, be families of subsets A_i of A resp. B_j of B . Then:

a). $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$ and $f(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} f(A_i)$.

b). $f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)$ and $f^{-1}(\bigcap_{i \in I} B_i) = \bigcap_{i \in I} f^{-1}(B_i)$.

0.A.11. Let A, I and J be sets. Then the map $f \mapsto (j \mapsto (i \mapsto f(i, j)))$ is a bijective map from $A^{I \times J}$ onto $(A^I)^J$.

0.A.12. Let I be a set. For a subset $J \in \mathfrak{P}(I)$, let $e_J: I \rightarrow \{0, 1\}$, $e_J(j) = 1$ if $j \in J$ and $e_J(i) = 0$ if $i \notin J$, denote the indicator function of J . Then:

a). The map $e: \mathfrak{P}(I) \rightarrow \{0, 1\}^I$ defined by $J \mapsto e_J$, is a bijective map from the power set of I onto the set $\{0, 1\}^I$ of all maps from I into $\{0, 1\}$.

b). Let I be a set. For a subset $J \in \mathfrak{P}(I)$, let J' be the complement of J in I . Prove the following equalities on indicator functions (by thinking these as functions with values in \mathbb{Z}):

$$e_{J \cap K} = e_J e_K, \quad e_{J \cup K} = e_J + e_K - e_J e_K, \quad e_{J-K} = e_J(1 - e_K).$$

In particular, $e_{J'} = 1 - e_J$ and $e_{J \Delta K} = e_J + e_K - 2e_J e_K$.

0.A.13. (Relations) Let X and Y be sets. A relation R between X and Y is a subset $R \subseteq X \times Y$, i.e. an element $R \in \mathfrak{P}(X \times Y)$. For the expression “ $(x, y) \in R$ ” we shall write “ xRy ” and say that “ x is related to y with respect to R ”, $x \in X$, $y \in Y$. In the case $Y = X$ we say that $R \subseteq X \times X$ is a relation on X .

a). The map $\mathfrak{P}(X \times Y) \rightarrow \mathfrak{P}(Y)^X$ defined by $R \mapsto (x \mapsto \{y \in Y \mid xRy\})$ is bijective. Write down the inverse of this map. (**Remark:** With this bijection, one can identify every relation $R \subseteq X \times Y$ between X and Y as a map from X into $\mathfrak{P}(Y)$.)

A relation $R \in \mathfrak{P}(X \times X)$ on X is called an equivalence relation if it satisfies:

- (1) (Reflexivity) : xRx for all $x \in X$.
- (2) (Symmetry) : xRy implies yRx , where $x, y \in X$.
- (3) (Transitivity): xRy and yRz implies xRz , where $x, y, z \in X$.

Let $\mathfrak{E}(X) \subseteq \mathfrak{P}(X \times X)$ denote the set of all equivalence relations on X .

b). The restriction of the map $\alpha : \mathfrak{P}(X \times X) \rightarrow \mathfrak{P}(\mathfrak{P}(X))$, $R \mapsto \{\{y \in X \mid xRy\} \mid x \in X\}$ is injective on the subset $\mathfrak{E}(X)$.

A partition \mathfrak{Z} of the set X is a subset $\mathfrak{Z} \subseteq \mathfrak{P}(X)$ of non-empty disjoint subsets of X such that their union is X . Therefore $\mathfrak{Z} \in \mathfrak{P}(\mathfrak{P}(X))$. Let $\mathfrak{Z}(X) \subseteq \mathfrak{P}(\mathfrak{P}(X))$ denote the set of all partitions of X .

c). In the situation of **b)**: α maps $\mathfrak{E}(X)$ bijectively onto $\mathfrak{Z}(X)$, i.e. to each equivalence relation R on X , α associates a unique partition $\alpha(R)$ of X and conversely.

The partition $\mathfrak{Z}(R)$ corresponding to the equivalence relation R on X is usually denoted by X/R and is called the quotient set of X with respect to the equivalence relation R . The elements $[x] := [x]_R := \{y \in X \mid xRy\} \in X/R$ are called the equivalence classes of x with respect to R .

d). Let $f : A \rightarrow B$ be a map. The relation \sim defined by $x \sim y$ if and only if $f(x) = f(y)$, is an equivalence relation on A . The equivalence classes with respect to \sim are precisely the *non-empty* fibres of f .

e). Give examples of relations which satisfy the two of the three properties of the equivalence relations, but not the third one. How many relations are there on the set with n elements and how many of them are reflexive resp. symmetric?

f). On the set \mathbb{N}^* of positive natural numbers, let \mid be the divisibility relation, i.e., $x \mid y$ if and only if x is a divisor of y . Show that \mid is an order on \mathbb{N}^* and 1 is the smallest element. The minimal elements (with respect to \mid) in $\mathbb{N}^* - \{1\}$ are precisely the prime numbers. Draw the Hasse-Diagramm for the set of divisors of 12 resp. of 30. The chains in \mathbb{N}^* with respect to \mid bijectively correspond to the finite or infinite sequences (q_0, q_1, q_2, \dots) with $q_n \in \mathbb{N}^*$ and $q_n \geq 2$ if $n \geq 1$. The corresponding chain $\{q_0, q_0q_1, q_0q_1q_2, \dots\}$ is maximal if and only if the sequence is infinite, $q_0 = 1$ and q_n are prime numbers.

g). Let \leq be a reflexive and transitive relation on the set A . Then the relation \sim defined by $a \sim b$ if and only if $a \leq b$ and $b \leq a$, is an equivalence relation on A . On the set \bar{A} of the equivalence classes of A with respect to \sim the relation defined by $[a] \leq [b]$ if and only if $a \leq b$, is a well-defined relation and is an order. (

Remark: It is to be shown in particular that the \leq -relationship for two equivalence classes does not depend on the representatives used for the definition. The problem to verify such independence from the choice of the representatives is typical for computation of equivalence classes.)

† **Georg Ferdinand Ludwig Philipp Cantor (1845-1918)** was born on 3 March 1845 in St Petersburg, Russia and died on 6 Jan 1918 in Halle, Germany. Georg Cantor's father, Georg Waldemar Cantor, was a successful merchant, working as a wholesaling agent in St Petersburg, then later as a broker in the St Petersburg Stock Exchange. Georg Waldemar Cantor was born in Denmark and he was a man with a deep love of culture and the arts. Georg's mother, Maria Anna Böhm, was Russian and very musical. Certainly Georg inherited considerable musical and artistic talents from his parents being an outstanding violinist. Georg was brought up a Protestant, this being the religion of his father, while Georg's mother was a Roman Catholic.

After early education at home from a private tutor, Cantor attended primary school in St Petersburg, then in 1856 when he was eleven years old the family moved to Germany. However, Cantor remembered his early years in Russia with great nostalgia and never felt at ease in Germany, although he lived there for the rest of his life and seemingly never wrote in the Russian language, which he must have known.

Cantor's father had poor health and the move to Germany was to find a warmer climate than the harsh winters of St Petersburg. At first they lived in Wiesbaden, where Cantor attended the Gymnasium, then they moved to Frankfurt. Cantor studied at the Realschule in Darmstadt where he lived as a boarder. He graduated in 1860 with an outstanding report, which mentioned in particular his exceptional skills in mathematics, in particular trigonometry. After attending the Höheren Gewerbeschule in Darmstadt from 1860 he entered the Polytechnic of Zurich in 1862. The reason Cantor's father chose to send him to the Höheren Gewerbeschule was that he wanted Cantor to become a shining star in the engineering firmament. However, in 1862 Cantor had sought his father's permission to study mathematics at university and he was overjoyed when eventually his father consented. His studies at Zurich, however, were cut short by the death of his father in June 1863. Cantor moved to the University of Berlin where he became friends with Herman Schwarz who was a fellow student. Cantor attended lectures by Weierstrass, Kummer and Kronecker. He spent the summer term of 1866 at the University of Göttingen, returning to Berlin to complete his dissertation on number theory *De aequationibus secundi gradus indeterminatis* in 1867.

While at Berlin Cantor became much involved with the Mathematical Society being president of the Society during 1864-65. He was also part of a small group of young mathematicians who met weekly in a wine house. After receiving his doctorate in 1867, Cantor taught at a girl's school in Berlin. Then, in 1868, he joined the Schellbach Seminar for mathematics teachers. During this time he worked on his habilitation and, immediately after being appointed to Halle in 1869, he presented his thesis, again on number theory, and received his habilitation.

At Halle the direction of Cantor's research turned away from number theory and towards analysis. This was due to Heine, one of his senior colleagues at Halle, who challenged Cantor to prove the open problem on the uniqueness of representation of a function as a trigonometric series. This was a difficult problem which had been unsuccessfully attacked by many mathematicians, including Heine himself as well as Dirichlet, Lipschitz and Riemann. Cantor solved the problem proving uniqueness of the representation by April 1870. He published further papers between 1870 and 1872 dealing with trigonometric series and these all show the influence of Weierstrass's teaching.

Cantor was promoted to Extraordinary Professor at Halle in 1872 and in that year he began a friendship with Dedekind who he had met while on holiday in Switzerland. Cantor published a paper on trigonometric series in 1872 in which he defined irrational numbers in terms of convergent sequences of rational numbers. Dedekind published his definition of the real numbers by "Dedekind cuts" also in 1872 and in this paper Dedekind refers to Cantor's 1872 paper which Cantor had sent him.

In 1873 Cantor proved the rational numbers countable, i.e. they may be placed in one-one correspondence with the natural numbers. He also showed that the algebraic numbers, i.e. the numbers which are roots of polynomial equations with integer coefficients, were countable. However his attempts to decide whether the real numbers were countable proved harder. He had proved that the real numbers were not countable by December 1873 and published this in a paper in 1874. It is in this paper that the idea of a one-one correspondence appears for the first time, but it is only implicit in this work.

A transcendental number is an irrational number that is not a root of any polynomial equation with integer coefficients. Liouville established in 1851 that transcendental numbers exist. Twenty years later, in this 1874 work, Cantor showed that in a certain sense 'almost all' numbers are transcendental by proving that the real numbers were not countable while he had proved that the algebraic numbers were countable.

Cantor pressed forward, exchanging letters throughout with Dedekind. The next question he asked himself, in January 1874, was whether the unit square could be mapped into a line of unit length with a 1-1 correspondence of points on each. In a letter to Dedekind dated 5 January 1874 he wrote: *Can a surface (say a square that includes the boundary) be uniquely referred to a line (say a straight line segment that includes the end points) so that for every point on the surface there is a corresponding point of the line and, conversely, for every point of the line there is a corresponding point of the surface? I think that answering this question would be no easy job, despite the fact that the answer seems so clearly to be "no" that proof appears almost unnecessary.*

The year 1874 was an important one in Cantor's personal life. He became engaged to Vally Guttmann, a friend of his sister, in the spring of that year. They married on 9 August 1874 and spent their honeymoon in Interlaken in Switzerland where Cantor spent much time in mathematical discussions with Dedekind. Cantor continued to correspond with Dedekind, sharing his ideas and seeking Dedekind's opinions, and he wrote to Dedekind in 1877 proving that there was a 1-1 correspondence of points on the interval $[0, 1]$ and points in p -dimensional space. Cantor was surprised at his own discovery and wrote: *I see it, but I don't believe it!*

Of course this had implications for geometry and the notion of dimension of a space. A major paper on dimension which Cantor submitted to Crelle's Journal in 1877 was treated with suspicion by Kronecker, and only published after Dedekind intervened on Cantor's behalf. Cantor greatly resented Kronecker's opposition to his work and never submitted any further papers to Crelle's Journal.

The paper on dimension which appeared in Crelle's Journal in 1878 makes the concepts of 1-1 correspondence precise. The paper discusses denumerable sets, i.e. those which are in 1-1 correspondence with the natural numbers. It studies sets of equal power, i.e. those sets which are in 1-1 correspondence with each other. Cantor also discussed the concept of dimension and stressed the fact that his correspondence between the interval $[0, 1]$ and the unit square was not a continuous map.

Between 1879 and 1884 Cantor published a series of six papers in *Mathematische Annalen* designed to provide a basic introduction to set theory. Klein may have had a major influence in having *Mathematische Annalen* published them. However there were a number of problems which occurred during these years which proved difficult for Cantor. Although he had been promoted to a full professor in 1879 on Heine's recommendation, Cantor had been hoping for a chair at a more prestigious university. His long standing correspondence with Schwarz ended in 1880 as opposition to Cantor's ideas continued to grow and Schwarz no longer supported the direction that Cantor's work was going. Then in October 1881 Heine died and a replacement was need to fill the chair at Halle.

Cantor drew up a list of three mathematicians to fill Heine's chair and the list was approved. It placed Dedekind in first place, followed by Heinrich Weber and finally Mertens. It was certainly a severe blow to Cantor when Dedekind declined the offer in the early 1882, and the blow was only made worse by Heinrich Weber and then Mertens declining too. After a new list had been drawn up, Wangerin was appointed but he never formed a close relationship with Cantor. The rich mathematical correspondence between Cantor and Dedekind ended later in 1882.

Almost the same time as the Cantor-Dedekind correspondence ended, Cantor began another important correspondence with Mittag-Leffler. Soon Cantor was publishing in Mittag-Leffler's journal *Acta Mathematica* but his important series of six papers in *Mathematische Annalen* also continued to appear. The fifth paper in this series *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* was also published as a separate monograph and was especially important for a number of reasons. Firstly Cantor realised that his theory of sets was not finding the acceptance that he had hoped and the *Grundlagen* was designed to reply to the criticisms. Secondly, The major achievement of the *Grundlagen* was its presentation of the transfinite numbers as an autonomous and systematic extension of the natural numbers.

Cantor himself states quite clearly in the paper that he realises the strength of the opposition to his ideas: ... *I realise that in this undertaking I place myself in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers.*

At the end of May 1884 Cantor had the first recorded attack of depression. He recovered after a few weeks but now seemed less confident. He wrote to Mittag-Leffler at the end of June: ... *I don't know when I shall return to the continuation of my scientific work. At the moment I can do absolutely nothing with it, and limit myself to the most necessary duty of my lectures; how much happier I would be to be scientifically active, if only I had the necessary mental freshness.*

At one time it was thought that his depression was caused by mathematical worries and as a result of difficulties of his relationship with Kronecker in particular. Recently, however, a better understanding of mental illness has meant that we can now be certain that Cantor's mathematical worries and his difficult relationships were greatly magnified by his depression but were not its cause. After this mental illness of 1884, he took a holiday in his favourite Harz mountains and for some reason decided to try to reconcile himself with Kronecker. Kronecker accepted the gesture, but it must have been difficult for both of them to forget their enmities and the philosophical disagreements between them remained unaffected.

Mathematical worries began to trouble Cantor at this time, in particular he began to worry that he could not prove the continuum hypothesis, namely that the order of infinity of the real numbers was the next after that of the natural numbers. In fact he thought he had proved it false, then the next day found his mistake. Again he thought he had proved it true only again to quickly find his error.

All was not going well in other ways too, for in 1885 Mittag-Leffler persuaded Cantor to withdraw one of his papers from *Acta Mathematica* when it had reached the proof stage because he thought it "... about one hundred years too soon". Cantor joked about it but was clearly hurt: *Had Mittag-Leffler had his way, I should have to wait until the year 1984, which to me seemed too great a demand! ... But of course I never want to know anything again about Acta Mathematica.*

Mittag-Leffler meant this as a kindness but it does show a lack of appreciation of the importance of Cantor's work. The correspondence between Mittag-Leffler and Cantor all but stopped shortly after this event and the flood of new ideas which had led to Cantor's rapid development of set theory over about 12 years seems to have almost stopped.

In 1886 Cantor bought a fine new house on Händelstrasse, a street named after the German composer Handel. Before the end of the year a son was born, completing his family of six children. He turned from the mathematical development of set theory towards two new directions, firstly discussing the philosophical aspects of his theory with many philosophers (he published these letters in 1888) and secondly taking over after Clebsch's death his idea of founding the *Deutsche Mathematiker-Vereinigung* which he achieved in 1890. Cantor chaired the first meeting of the Association in Halle in September 1891, and despite the bitter antagonism between himself and Kronecker, Cantor invited Kronecker to address the first meeting.

Kronecker never addressed the meeting, however, since his wife was seriously injured in a climbing accident in the late summer and died shortly afterwards. Cantor was elected president of the *Deutsche Mathematiker-Vereinigung* at the first meeting and held this post until 1893. He helped to organise the meeting of the Association held in Munich in September 1893, but he took ill again before the meeting and could not attend.

Cantor published a rather strange paper in 1894 which listed the way that all even numbers up to 1000 could be written as the sum of two primes. Since a verification of Goldbach's conjecture up to 10000 had been done 40 years before, it is likely that this strange paper says more about Cantor's state of mind than it does about Goldbach's conjecture.

His last major papers on set theory appeared in 1895 and 1897, again in *Mathematische Annalen* under Klein's editorship, and are fine surveys of transfinite arithmetic. The rather long gap between the two papers is due to the fact that although Cantor finished writing the second part six months after the first part was published, he hoped to include a proof of the continuum hypothesis in the second part. However, it was not to be, but the second paper describes his theory of well-ordered sets and ordinal numbers.

In 1897 Cantor attended the first International Congress of Mathematicians in Zurich. In their lectures at the Congress: ... *Hurwitz openly expressed his great admiration of Cantor and proclaimed him as one by whom the theory of functions has been enriched. Jacques Hadamard expressed his opinion that the notions of the theory of sets were known and indispensable instruments.*

At the Congress Cantor met Dedekind and they renewed their friendship. By the time of the Congress, however, Cantor had discovered the first of the paradoxes in the theory of sets. He discovered the paradoxes while working on his survey papers of 1895 and 1897 and he wrote to Hilbert in 1896 explaining the paradox to him. Burali-Forti discovered the paradox independently and published it in 1897. Cantor began a correspondence with Dedekind to try to understand how to solve the problems but recurring bouts of his mental illness forced him to stop writing to Dedekind in 1899.

Whenever Cantor suffered from periods of depression he tended to turn away from mathematics and turn towards philosophy and his big literary interest which was a belief that Francis Bacon wrote Shakespeare's plays. For example in his illness of 1848 he had requested that he be allowed to lecture on philosophy instead of mathematics and he had begun his intense study of Elizabethan literature in attempting to prove his Bacon-Shakespeare theory. He began to publish pamphlets on the literary question in 1896 and 1897. Extra stress was put on Cantor with the death of his mother in October 1896 and the death of his younger brother in January 1899. In October 1899 Cantor applied for, and was granted, leave from teaching for the winter semester of 1899-1900. Then on 16 December 1899 Cantor's youngest son died. From this time on until the end of his life he fought against the mental illness of depression. He did continue to teach but also had to take leave from his teaching for a number of winter semesters, those of 1902-03, 1904-05 and 1907-08. Cantor also spent some time in sanatoria, at the times of the worst attacks of his mental illness, from 1899 onwards. He did continue to work and publish on his Bacon-Shakespeare theory and certainly did not give up mathematics completely. He lectured on the paradoxes of set theory to a meeting of the Deutsche Mathematiker-Vereinigung in September 1903 and he attended the International Congress of Mathematicians at Heidelberg in August 1904.

In 1905 Cantor wrote a religious work after returning home from a spell in hospital. He also corresponded with Jourdain on the history of set theory and his religious tract. After taking leave for much of 1909 on the grounds of his ill health he carried out his university duties for 1910 and 1911. It was in that year that he was delighted to receive an invitation from the University of St Andrews in Scotland to attend the 500th anniversary of the founding of the University as a distinguished foreign scholar. The celebrations were 12-15 September 1911 but during the visit he apparently began to behave eccentrically, talking at great length on the Bacon-Shakespeare question; then he travelled down to London for a few days.

Cantor had hoped to meet with Russell who had just published the Principia Mathematica. However ill health and the news that his son had taken ill made Cantor return to Germany without seeing Russell. The following year Cantor was awarded the honorary degree of Doctor of Laws by the University of St Andrews but he was too ill to receive the degree in person.

Cantor retired in 1913 and spent his final years ill with little food because of the war conditions in Germany. A major event planned in Halle to mark Cantor's 70th birthday in 1915 had to be cancelled because of the war, but a smaller event was held in his home. In June 1917 he entered a sanatorium for the last time and continually wrote to his wife asking to be allowed to go home. He died of a heart attack.

Hilbert described Cantor's work as: *...the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity.*