Algebra, Arithmetic and Geometry – With a View Toward Applications / 2005

Lectures : Tuesday/Thursday 18:15–19:15 ; LH-1, Department of Mathematics

0.B Finite Sets

0.B.1. Let X be a finite set with n elements. For $i \in \mathbb{N}$, let $\mathfrak{P}_i(X)$ be the set of all subsets Y of X with $|Y| = i$. Show that: If $i \in \mathbb{N}$ with $0 \le i < n/2$ (resp. with $n/2 < i \le n$), then there exists an injective map $f_i : \mathfrak{P}_i(X) \to \mathfrak{P}_{i+1}(X)$ such that $Y \subseteq f_i(Y)$ for all $Y \in \mathfrak{P}_i(X)$ (resp. an injective map $g_i : \mathfrak{P}_i(X) \to \mathfrak{P}_{i-1}(X)$ such that $g_i(Y) \subseteq Y$ for all $Y \in \mathfrak{P}_i(X)$). (**Hint :** Let $0 \le i < n/2$. A pair $(Y, Y') \in \mathfrak{P}_i(X) \times \mathfrak{P}_{i+1}(X)$ is called *amicable* if $Y \subseteq Y'$. Let $\mathfrak{R}$ be a subset of $\mathfrak{P}_i(X)$ with $|\mathfrak{R}| =: r$. Further, let $\mathfrak{R}'$ be the set of all those $Y' \in \mathfrak{P}_{i+1}(X)$ which are amicable to at least one $Y \in \mathfrak{R}$. Put $s := |\mathfrak{R}'|$. Then $r(n-i) \le s(i+1)$ and hence $r \le s$. Now use the Marriage-theorem see Exercise (1.2).)

0.B.2. Let $X_1, \ldots, X_n$ be finite sets. For $J \subseteq \{1, \ldots, n\}$, let $X_J := \bigcap_{i\in J} X_i$ with $X_\emptyset := \bigcup_{i=1}^n X_i$. Generalize the formula $|Y \cup Z| = |Y| + |Z| - |Y \cap Z|$ for finite sets Y, Z, prove the well-known Sylvester's (Sieve–) formula[1]):

$$\sum_{J\in\mathfrak{P}(\{1,\ldots,n\})} (-1)^{|J|}|X_J| = 0\,, \quad \text{i.e.} \quad |X| = \sum_{\emptyset\neq J\in\mathfrak{P}(\{1,\ldots,n\})} (-1)^{|J|-1}|X_J|\,.$$

(**Hint :** By induction on n. — Variant : For $k = 1, \ldots, n$, let Y_k be the set of elements $x \in X_\emptyset$ which belong to exactly k of the sets $X_1, \ldots, X_n$. Then Y_k, $1 \le k \le n$ are pairwise disjoint. Using Exercise T2.2 b) show that

$$\sum_{\substack{J\in\mathfrak{P}(\{1,\ldots,n\})\\ |J|\ \text{even}}} |X_J| = \sum_{k=1}^n 2^{k-1}|Y_k| = \sum_{\substack{J\in\mathfrak{P}(\{1,\ldots,n\})\\ |J|\ \text{odd}}} |X_J|\,.)$$

0.B.3. a). Let X be a finite set with m elements. Let p_m denote the number of permutations of X which donot have fixed points and let $s_m = m!$ be the number of all all permutations of X. Show that:

$$\frac{p_m}{s_m} = \frac{1}{0!} - \frac{1}{1!} + \cdots + (-1)^m \cdot \frac{1}{m!}\,.$$

(**Hint :** Let $X = \{x_1, \ldots, x_m\}$. Set $X_i := \{\sigma \in \mathfrak{S}(X) : \sigma(x_i) = x_i\}$ and compute $s_m - p_m = |\bigcup_{i=1}^m X_i|$ using the Sieve formula in Exercise 2.2. — **Remark :** Note that $\lim_{m\to\infty}(p_m/s_m) = e^{-1}$, where $e = 2,718\ldots$ is the base of the natural logarithm.) — The number of permutations of X with exactly r fixed points is $\binom{m}{r} p_{m-r}$, $0 \le r \le m$. (Proof!)

b). Let X be a finite set with m elements and let Y be a finite set with n elements. The number of surjective maps from X in Y is

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^n\binom{n}{n}(n-n)^m\,.$$

(**Hint :** Let $Y = \{y_1, \ldots, y_n\}$. Set $P_i := \{f \in Y^X : y_i \notin \operatorname{im} f\}$ and compute the number $|\bigcup_{i=1}^n P_i|$ of non-surjective maps using the Sieve formula in Exercise 2.2.)

0.B.4. Let I be a finite index set with n elements and let $\sigma_i \in \mathbb{N}$ for $i \in I$, $\pi := \prod_{i\in I}\sigma_i$, $\sigma := \sum_{i\in I}\sigma_i$ and $\sigma_H := \sum_{i\in H}\sigma_i$ for $H \subseteq I$. Then

$$\sum_{H\subseteq I}(-1)^{|H|}\binom{\sigma_H}{n} = (-1)^n\pi \quad \text{and} \quad \sum_{H\subseteq I}(-1)^{|H|}\binom{\sigma_H}{n+1} = \frac{(-1)^n}{2}(\sigma - n)\pi\,,$$

(**Hint :** Let $X = \bigcup_{i\in I} X_i$, where X_i are pairwise disjoint subsets with $|X_i| = \sigma_i$. For a proof of the first formula consider the set $\mathfrak{P}_n(X)$ and its subsets $Y_i := \{A \in \mathfrak{P}_n(X) \mid A \cap X_i = \emptyset\}$ and use the Sieve formula in Exercise 2.2 to find $|\bigcup_{i\in I} Y_i|$.)

On the other side one can see (simple) test-exercises.

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[1]) This is also called the Inclusion-Exclusion principle

Test-Exercises

T0.B.1. Let X be a finite set with n elements.

a). The number of subsets of X is 2^n (Induction).

b). If $n \in \mathbb{N}^*$, then the number of subsets of X with an even number of elements is equal to the number of subsets of X with an odd number of elements. Moreover, this number is equal to 2^{n-1}. (**Hint:** Let $a \in X$. The map defined by $A \mapsto A \cup \{a\}$, if $a \notin A$, resp. $A \setminus \{a\}$, if $a \in A$, is a bijective map from the set of subsets with an even number of elements onto the set of subsets with an odd number of elements.)

T0.B.2. (F a c t o r i a l s a n d B i n o m i a l c o - e f f i c i e n t s) For natural number $n \in \mathbb{N}$, the natural number $n! := \prod_{i=1}^{n} i 1 \cdot 2 \cdots n$ is called the n-f a c t o r i a l. Note that $0! = 1$.

Let X and Y be two finite sets. Then:

a). If $|X| = m \leq n = |Y|$, then the set $\mathfrak{I}(X, Y) := \{f : X \to Y \mid f \text{ is injective }\}$ has cardinality $n!/(n-m)!$.

b). The set of all permutations $\mathfrak{S}(X)$ has exactly $|X|!$ elements.

c). For any $m \in \mathbb{N}$ with $m \leq |Y|$, the set $\mathfrak{P}_m(Y) := \{Z \in \mathfrak{P}(Y) \mid |Z| = m\}$ has cardinality $n!/(n-m)!$.

(**Remark:** For $0 \leq m \leq n$, the natural numbers $\binom{n}{m} := n!/m! \cdot (n-m)!$ are called b i n o m i a l c o - e f f i c i e n t s, since they occur in the Binomial theorem. For $m, n \in \mathbb{Z}$, we define $\binom{n}{m} := 0$ if either $m < 0$ or if $n < m$.)

d). Prove the formuals: $\binom{n}{m} = \binom{n}{n-m}$ and $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$. (**Hint:** For the first let $0 \leq m \leq n$. Then the map $\mathfrak{P}_m(\{1, \ldots, n\}) \to \mathfrak{P}_{n-m}(\{1, \ldots, n\})$ defined by $J \mapsto \{1, \ldots, n\} \setminus J$ is a bijective map. — For the second, let $1 \leq m \leq n$ and let $\mathfrak{X} := \{I \in \mathfrak{P}_m(\{1, \ldots, n+1\}) \mid n+1 \notin I\}$ and $\mathfrak{Y} := \{I \in \mathfrak{P}_m(\{1, \ldots, n+1\}) \mid n+1 \in I\}$. Then the maps $\mathfrak{X} \to \mathfrak{P}_m(\{1, \ldots, n\})$ defined by $J \mapsto J$ and $\mathfrak{Y} \to \mathfrak{P}_{m-1}(\{1, \ldots, n\})$ defined by $J \mapsto J \setminus \{n+1\}$ are bijective and hence $|\mathfrak{X}| = \binom{n}{m}$, $|\mathfrak{Y}| = \binom{n}{m-1}$ and $\{\mathfrak{X}, \mathfrak{Y}\}$ is a partition of $\mathfrak{P}_m(\{1, \ldots, n, n+1\})$.)

T0.B.3. a). From 1a) deduce that: For $n \in \mathbb{N}$, $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

b). From 1b) deduce that: For $n \in \mathbb{N}^*$, $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0$.

c). Let X be a finite set with n elements. The number of pairs (X_1, X_2) in $\mathfrak{P}(X) \times \mathfrak{P}(X)$ with $X_1 \cap X_2 = \emptyset$ is 3^n (Induction). General: The number of r–tuples $(X_1, \ldots, X_r)$ of pairwise disjoint subsets $X_1, \ldots, X_r \subseteq X$ is equal $(r+1)^n$, $r \in \mathbb{N}$.

d). For $m, n, k \in \mathbb{N}$, $\binom{m+n}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0}$. In particular, $\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$ for $n \in \mathbb{N}$. (**Hint:** Let X, Y be disjoint sets with $|X| = m$, $|Y| = n$. The assignment $A \mapsto (A \cap X, A \cap Y)$ defines a bijective map $\mathfrak{P}(X \cup Y) \to \mathfrak{P}(X) \times \mathfrak{P}(Y)$.)

T0.B.4. Let m be a natural number (resp. a positive natural number) and let n be another natural number. Let $\mathrm{a}(m, n)$ (resp. $\mathrm{b}(m, n)$) denote the number of m–tuples $(x_1, \ldots, x_m) \in \mathbb{N}^m$ with $x_1 + \cdots + x_m \leq n$ (resp. $x_1 + \cdots + x_m = n$). Show that

$$\mathrm{a}(m, n) = \binom{n+m}{m}, \quad \mathrm{b}(m, n) = \binom{n+m-1}{m-1}.$$

(**Hint:** Note that $\mathrm{a}(m-1, n) = \mathrm{b}(m, n)$ and $\mathrm{a}(m, n) = \mathrm{a}(m, n-1) + \mathrm{a}(m-1, n)$ if $m \geq 1$ and use induction on $n+m$. —Variant: The map $(x_1, \ldots, x_m) \mapsto \{x_1+1, \ x_1+x_2+2, \ldots, x_1+\cdots+x_m+m\}$ maps the set of m–tuples $(x_1, \ldots, x_m) \in \mathbb{N}^m$ with $x_1 + \cdots + x_m \leq n$ bijectively onto the set of m–element subsets of $\{1, 2, \ldots, n+m\}$.)

T0.B.5. Let $\mathfrak{X} = (X_1, \ldots, X_r)$ and let $\mathfrak{Y} = (Y_1, \ldots, Y_r)$ be partitions of the set X into r pairwise disjoint subsets each of them with $n \geq 1$ elements (i.e. $\bigcup_{i=1}^{r} X_i = X$ and $X_i \cap X_j = \emptyset$ for $i \neq j$ and analogously for $\mathfrak{Y}$). Show that: $\mathfrak{X}$ and $\mathfrak{Y}$ has a common representative system, i.e. there exist r distinct elements $x_1, \ldots, x_r$ in X such that each x_i belongs to exactly one of the subset $X_1, \ldots, X_r$ and exactly one of the subset $Y_1, \ldots, Y_r$. (**Hint:** Using the Marriage-theorem find a permutation $\sigma \in \mathfrak{S}_r$ such that $X_i \cap Y_{\sigma(i)} \neq \emptyset$ for every $1 \leq i \leq r$.)