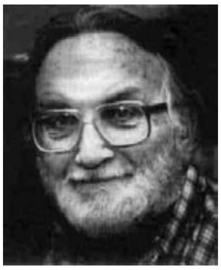
Algebra, Arithmetic and Geometry – With a View Toward Applications / 2005
Lectures : Tuesday/Thursday 18:15–19:15 ; LH-1, Department of Mathematics

0.C Ordered Sets



Max August Zorn†
(1906-1993)

0.C.1. Let $(X, \leq)$ be an ordered set in which every subset has a least upper bound and has a greatest lower bound. Further, let $f : X \to X$ be an increasing map and let F be the fixed points of f. Show that:

a). If $\{x \in X \mid f(x) < x\} \neq \emptyset$ and if a is its greatest lower bound, then either $a \in F$ or $f(a) \in F$.

b). If $\{x \in X \mid x < f(x)\} \neq \emptyset$ and if z is its least upper bound, then either $z \in F$ or $f(z) \in F$.

c). F is non-empty. Further, the least upper bound and the greatest lower bound of F belong to F.

0.C.2. a). (Lattice) An ordered set $(X, \leq)$ is called a lattice if for every two elements x, y in X, $\operatorname{Sup}\{x, y\}$ and $\operatorname{Inf}\{x, y\}$ exist. The power set $\mathfrak{P}(X), \subseteq)$ of a set with respect to the natural inclusion is a lattice. Every totally ordered set is a lattice. If $X_i, i \in I$, is a family of lattices, then so is the product set $\prod_{i \in I} X_i$ with respect to the product order. If X is a lattice, then X is also lattice with respect to the inverse order; this lattice is called the (dual lattice).

b). Let $(X, \leq)$ be a lattice with largest and smallest element. Suppose that X has the following property : if $x, y \in X$ and if $\operatorname{Sup}\{x, y\}$ is a *direct successer* [1]) of x, then $\operatorname{Inf}\{x, y\}$ is a *direct predecesser* of y. Prove the following Chain theorem: *If X has a finite maximal chain, then every chain in X is finite and the lengths* [2]) *of all maximal chains in X are equal.* (**Hint :** It is enough to prove that: if X has a finite maximal chain $x_0 < \cdots < x_n$ of length n, then every finite chain $y_0 < \cdots < y_m$ in X has length $m \leq n$. Induction on n: if $n \geq 1, m \geq 1$, then apply induction hypothesis on the lattice $\{x \in M : x \leq x_{n-1}\}$. In the case $y_{m-1} \not\leq x_{n-1}$ consider the element $\operatorname{Inf}\{y_{m-1}, x_{n-1}\}$. This element is $\leq x_{n-1}$ and is a direct predecesser of y_{m-1}. If $y_{m-2} \not\leq \operatorname{Inf}\{y_{m-1}, x_{n-1}\}$, then consider $\operatorname{Inf}\{y_{m-2}, \operatorname{Inf}\{y_{m-1}, x_{n-1}\}\}$ and so on... (Induction on m). Note that x_0 resp. x_n is the smallest resp. greatest element in X.) — Give an example of a (with 5 elements) in which there are maximal chains of different lengths.

0.C.3. (Dedekind's Chain Theorem) Suppose that X is artinian and noetherian and that X has the following property: if $x, y \in X$ and if $\operatorname{Sup}\{x, y\}$ is a direct successer of x and y, then $\operatorname{Inf}\{x, y\}$ is a direct predecesser of x and y. Show that: *all maximal chains in X have the same (finite) lengths.* (**Hint :** Similar to that of part a).)

0.C.4. a). (Dilworth's Theorem) Let $(X, \leq)$ be a finite ordered set and let m be the cardinality of a largest possible *anti-chain* [3]) in X. Show that X can be partitioned into m chains and X cannot be partitioned into r chains with $r < m$. — This natural number m is called the Dilworth's number of X. (**Proof.** By induction on the cardinality of X. Let $Y \subseteq X$ be an anti-chain in X of cardinality m. Let $A := \{a \in X \mid a < y \text{ for some } y \in Y\}$ and let $B := \{b \in X \mid y < b \text{ for some } y \in Y\}$. Then $X = A \uplus B \uplus Y$.

[1]) Let a, b be two elements in an ordered set $(X, \leq)$. If $a < b$ and $(a, b) := \{x \in X \mid a < x < b\} = \emptyset$, then we say that b is a direct successor of a and we say that a is a direct predecessor of b.

[2]) By the length of a finite chain C in an ordered set $(X, \leq)$, we mean $|C| - 1$.

[3]) An anti-chain in an ordered set $(X,\leq)$ is a subset of X consisting of pairwise uncomparable elements.

We divide the proof in the following four cases. **i).** $A = \emptyset = B$. **ii).** $A \neq \emptyset$, $B \neq \emptyset$. **iii).** $A \neq \emptyset$, $B = \emptyset$. **iv).** $A = \emptyset$, $B \neq \emptyset$. The proof in the case i) is trivial. For case ii) : Put $X_A := Y \cup A$ and $X_B := Y \cup B$. Then by induction both X_A and X_B can be partitioned into m chains. Then X can also be partitioned into m chains. For the case iii) : Let $y \in Y$ and let C be a chain of maximal length with $y \in C$. Let $X' := X \setminus C$. Note that the extremal elements of C are also extremal elements of X. Let m' denote the the maximal number of pairwise uncomparable elements in X'. Then $m - 1 \leq m' \leq m$, since $Y \setminus \{y\} \subseteq X'$. Choose an anti-chain Y' in X' with $\operatorname{card}(Y') = m'$. We now further consider the following two cases : iii.a) : $m' = m$. In this case replace Y by Y' and then apply the case ii) to complete the proof. iii.b) : $m' = m - 1$. In this case apply induction hypothesis to X' to complete the proof. Proof in the case iv) is similar to that of case iii).)

b). (E. Sperner) Let X be a finite set with n elements. Show that the Dilworth's number of the ordered set $(\mathfrak{P}(X), \subseteq)$ is $\binom{n}{[n/2]}$, where $[n/2]$ is the integral part of $n/2$. (**Hint :** Use the maps f_i , $0 \leq i < n/2$ and g_i , $n/2 < i \leq n$ of Exercise 2.1 to give an explicit partition of $\mathfrak{P}(X)$ into $\binom{n}{[n/2]}$ chains. **Variant :** if $\mathfrak{S} \subseteq \mathfrak{P}(X)$ be an anti-chain in $\mathfrak{P}(X)$ then $|\mathfrak{S}| \leq \binom{n}{[n/2]}$ as follows : For $Y \in \mathfrak{S}$, let $\mathfrak{C}_Y$ be the set of all maximal chains in $\mathfrak{P}(X)$ in which Y appears as an element. Then $|\mathfrak{C}_Y| = (n - |Y|)! \cdot |Y|!$ and $\mathfrak{C}_Y \cap \mathfrak{C}_Z = \emptyset$ if $Y, Z \in \mathfrak{S}$, $Y \neq Z$. Since there are $n!$ maximal chains, it follows that $\sum Y \in \mathfrak{S}(n - |Y|)! \cdot |Y|! \ \leq n!$ and so $|\mathfrak{S}|) \leq \binom{n}{[n/2]}$.) For $1 \leq i \leq n$, what is the Dilworth's number of $\mathfrak{P}_{\leq i}(X) := \{Y \in \mathfrak{P}(X) \mid |Y| \leq i\}$?

0.C.5. (Filters) Let X be a set. A filter $\mathfrak{F}$ on X is a subset of $\mathfrak{P}(X)$ with the following properties : (1) The intersection of finitely many elements of $\mathfrak{F}$ is again an element of $\mathfrak{F}$. (2) If $Y \in \mathfrak{F}$ and if $Y \subseteq Z \subseteq X$, then $Z \in \mathfrak{F}$. Note that for every filter $\mathfrak{F}$ on X , the set $X \in \mathfrak{F}$ by (1), since the intersection over the empty family is the set X . A filter $\mathfrak{F}$ on X is equal to $\mathfrak{P}(X)$ if and only if $\emptyset \in \mathfrak{F}$. [†] A filter basis $\mathfrak{B}$ on X is a subset of $\mathfrak{P}(X)$ satisfying the following property : (1′) The intersection of finitely many elements of $\mathfrak{B}$ contains an element of $\mathfrak{B}$. — A filter $\mathfrak{F}$ on X is called fixed if the intersection of all its elements is non-empty, otherwise is called free. The set of all filters on X is ordered by the natural inclusion. Maximal elements in the set of filters different from $\mathfrak{P}(X)$ are called ultra filters.

a). Give all filters on a finite set.

b). If X is not a finite set, then the complements of the finite subsets of X form a free filter diferent from $\mathfrak{P}(X)$ on X . (**Remark :** This is called the (well-known) Fréchet–filter.)

c). If $\mathfrak{B}$ is a filter basis, then the set of all $Y \subseteq X$ which contain a subset $Z \in \mathfrak{B}$, is a filter on X . (this filter is called the filter generated by $\mathfrak{B}$. If $\mathfrak{S} \subseteq \mathfrak{P}(X)$, the the set of all intersections of each finite family of elements from $\mathfrak{S}$ is a filter–basis.

d). If $X \neq \emptyset$, then the set of filters diferent from $\mathfrak{P}(X)$ on X is inductively ordered (with respect to the natural inclusion). Every filter different from $\mathfrak{P}(X)$ on a set X is contained in an ultra–filter.

e). If $\mathfrak{F}$ is a fixed ultra–filter on X , then $\mathfrak{F} = \{Y \subseteq Y \mid x \in Y\}$ for some $x \in X$. If X is not finite, then there exists free ultra–filter on X .

f). A filter $\mathfrak{F}$ different from $\mathfrak{P}(X)$ on X is an ultra-filter if and only if for every $Y \subseteq X$ we have: $Y \in \mathfrak{F}$ or $(X \setminus Y) \in \mathfrak{F}$. (**Hint :** Consider $\{Y \cap Z : Z \in \mathfrak{F}\}$.) On a not finite set X , the Fréchet–filter is not an ultra–filter.

0.C.6. (Modular Lattice) Let V be a lattice (see Exercise 5). In lattice theory, it is customarey to denote $x \sqcup y := \operatorname{Sup}\{x, y\}$ and $x \sqcap y := \operatorname{Inf}\{x, y\}$ for $x, y \in V$ and more generally $\sqcup_{i \in I} x_i := \operatorname{Sup}\{x_i : i \in I\}$ respectively, $\sqcap_{i \in I} x_i := \operatorname{Inf}\{x_i : i \in I\}$ for a family $x_i, i \in I$ of elements in V . Suppose that V has a greatest as well as smallest element, then the above elements exist if the indexed set I is finite. — V is called modular, if for all $x, y, z \in V$ with $z \leq x$ we have : $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup z$ (Modular laws). Suppose that V is modular and $u, v \in V$. Show that : the maps $x \mapsto x \sqcap v$ and $y \mapsto y \sqcup u$ are inverse isomorphisms of each other of the ordered intervals $[u, u \sqcup v]$ and $[u \sqcap v, v]$. — Deduce that : A modular lattice with greatest and smallest elements satisfies the condition which is used for the proving the chain theorem given in the Exercise 0.C.2 -a).

[†] Note that some authors donot accept that $\mathfrak{P}(X)$ is a filter on X . They further assume that $\emptyset \notin \mathfrak{F}$ for every filter $\mathfrak{F}$ on X .

0.C.7. Let I be a well–ordered set and let $(X_i)_{i\in I}$ be a family of ordered sets. For two elements $(x_i)_{i\in I}, (y_i)_{i\in I} \in X := \prod_{i\in I} X_i$ we define $(x_i) \leq (y_i)$ if and only if either $(x_i) = (y_i)$ or $(x_i) \neq (y_i)$ and for the smallest index i_0 in the set $\{i \in I : x_i \neq y_i\}$ we have $x_{i_0} < y_{i_0}$. Show that the relation $\leq$ define an order on X. (**Remark:** This order is called the lexicographic order on X.) If $X_i \neq \emptyset$ for all $i \in I$, then X is totally ordered if and only if all $X_i, i \in I$, are totally ordered. If $X_i \neq \emptyset$ for all $i \in I$, then X is well–ordered if and only if all $X_i, i \in I$, are well–ordered and for almost all (i. e., for all but finitely many) indices $i \in I$, the sets X_i are singletons.

0.C.8. Let $(X, \leq)$ be an ordered set. A section of X is a subset A of X with the following property: if $x \in A$, $y \in X$ with $y \leq x$, then $y \in A$. — The subsets $\emptyset$ and X are sections of X. Arbitrary intersections and arbitrary unions of sections of X are again sections of X. For $a \in X$, the subsets $A_a := \{x \in X \mid x < a\}$ and $\overline{A}_a := \{x \in X \mid x \leq a\}$ are sections of X.

a). The map $a \mapsto \overline{A}_a$ from X into $\mathfrak{P}(X)$ is a strictly monotone increasing (where $\mathfrak{P}(X)$ is ordered by the natural inclusion) and induces an isomorphism of X onto a subset of $\mathfrak{P}(X)$.

b). Suppose further that $(X, \leq)$ is well-ordered. Then show that: If A is a section of X, $A \neq X$, then there exists exactly one $a \in X$ such that $A = A_a$. The map $a \mapsto A_a$ is an isomorphism of X onto the set of sections different from X which is ordered by the natural inclusion. The set of sections of X is well-ordered and has a greatest element.

0.C.9. Let X be a well–ordered set and let g be strictly increasing map from X into itself. Show that:

a). $x \leq g(x)$ for all $x \in X$.

b). If $\operatorname{im} g$ is a section of X, then $g = \mathrm{id}_X$. In particular, id_X is the only isomorphism of X onto itself.

c). If X and Y are two well–ordered sets, then there is atmost one isomorphism of X onto a section of Y.

0.C.10. (Irreducible Elements) Let V be a lattice with greatest and least elements. We used the notation introduced in Exercise ??. An element $z \in V$ is called irreducible (with respect to $\sqcap$), if z is not the greatest element of V and if $z = x \sqcap y$ with $x, y \in V$ implies either $z = x$ or $z = y$. A representation of the form $z = \sqcap_{i\in I} x_i$ with irreducible elements $x_i \in V$ is called a decomposition of z into irreducible elements (with respect to $\sqcap$). This decompsition is called irredundant, if $z \neq \sqcap_{i\in I\smallsetminus\{r\}} x_i$ for every $r \in I$.

a). Suppose that V is noetherian. Using noetherian induction show that: every element $x \in V$ has a decomposition into finitely many irreducible elements and hence also has an irredundant decomposition into finitely many irreducible elements.

b). Suppose that V is distributive, i. e., $x\sqcup(y\sqcap z) = (x\sqcup y)\sqcap(x\sqcup z)$ and $x\sqcap(y\sqcup z) = (x\sqcap y)\sqcup(x\sqcap z)$ for all $x, y, z \in V$. Show that: If $x = \sqcap_{i\in I} x_i$ and $x = \sqcap_{j\in J} x'_j$ are two irredundant decompositions of an element $x \in V$ in finitely many irreducible elements, then $|I| = |J|$, and there exists a bijective map $\sigma : I \to J$ such that $x_i = x'_{\sigma(i)}$ for all $i \in I$.

c). Prove the following Exchange theorem: *Let V be a modular* (see Exercise ??). *If $x = \sqcap_{i\in I} x_i$ and $x = \sqcap_{j\in J} x'_j$ are two irredundant decompositions of $x \in V$ into finitely many irreduzible elements and if $r \in I$, then there exists an element $s \in J$ such that $x = x'_s \sqcap \sqcap_{i\in I\smallsetminus\{r\}} x_i$. This new decomposition of x is irrendundant.* Deduce that in particular, $|I| = |J|$. (**Hint:** For the proof of Exchange theorem: Let $\widetilde{x} := \sqcap_{i\in I\smallsetminus\{r\}} x_i$. For arbitrary $u, w \in V$ with $x = \widetilde{x} \sqcap x_r \leq u$, from the modular laws, we have the following chain of equalities:

$$\begin{aligned}((\widetilde{x} \sqcap u) \sqcup x_r) \sqcap ((\widetilde{x} \sqcap w) \sqcup x_r) &= (((\widetilde{x} \sqcap u) \sqcup x_r) \sqcap (\widetilde{x} \sqcap w)) \sqcup x_r \\ &= ((((\widetilde{x} \sqcap u) \sqcup x_r) \sqcap \widetilde{x}) \sqcap w) \sqcup x_r \\ &= (((\widetilde{x} \sqcap u) \sqcup (x_r \sqcap \widetilde{x})) \sqcap w) \sqcup x_r \\ &= (\widetilde{x} \sqcap u \sqcap w) \sqcup x_r\,;\end{aligned}$$

and hence $sqcap_{j\in J}((\widetilde{x} \sqcup x'_j) \sqcup x_r) = (\widetilde{x} \sqcap\ \sqcap_{j\in J} x'_j) \sqcup x_r = x_r$. Therefore $(\widetilde{x} \sqcap x'_s) \sqcup x_r = x_r$ for a $s \in J$.)

d). Define irreducible elements with respect to $\sqcup$ and formulate the analogous assertions for them.

Below one can see (simple) test-exercises.

Test-Exercises

T0.C.1. a). Let $(X, \leq)$ be a finite non-empty ordered set. Show that X has (at least one) a minimal and (at least one) a maximal element.

b). Let $(X, \leq)$ be a finite totally ordered set with n elements. Show that there exists exactly one isomorphism of the interval $[1, n] = \{1, \ldots, n\} \subseteq \mathbb{N}$ (with the natural order) onto X.

c). Let $(X, \leq)$ be an inductively ordered set and let $x \in X$. Show that there is a maximal element $z \in X$ such that $x \leq z$. (Apply *Zorn's* † *lemma* [4]) to the subset $\{y \in X \mid x \leq y\}$.)

T0.C.2. Let X be a set. Then the power set $\mathfrak{P}(X)$ of X is (with respect to the natural inclusion) noetherian resp. artinian if and only if X is finite.

T0.C.3. Let $(X, \leq)$ be a well-ordered set. Suppose that every element in X which is not the largest element in X has a direct successer in M. Is it true that every element in X which is not the smallest element in X necessarily have a direct predesser in X ?

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† **Max August Zorn (1906-1993)** Max Zorn was born on 6 June 1906 in Krefeld, Germany and died on 9 March 1993 in Bloomington, Indiana, USA. Max Zorn was born in Krefeld in western Germany, about 20 km northwest of Dusseldorf. He attended Hamburg University where he studied under Artin. Hamburg was Artin's first academic appointment and Zorn became his second doctoral student. He received his Ph.D. from Hamburg in April 1930 for a thesis on alternative algebras. His achievements were considered outstanding by the University of Hamburg and he was awarded a university prize. He was appointed as an assistant at Halle but he did not have the opportunity to work there for long since, in 1933, he was forced to leave Germany because of the Nazi policies. He was not, however, Jewish. Zorn emigrated to the United States and was appointed a Sterling Fellow at Yale University. He worked there from 1934 to 1936 and it was during this period that he proposed "Zorn's Lemma" for which he is best known. Since Zorn is best known for "Zorn's Lemma" it is perhaps appropriate that we should begin a discussion of his mathematical achievements by considering this contribution. Of course Zorn did not call his result "Zorn's Lemma", rather it was given by him as a "maximum principle" in a short paper entitled : *A remark on method in transfinite algebra*, which he published in the Bulletin of the American Mathematical Society in 1935. Perhaps in passing we should note that the name "Zorn's Lemma" was due to John Tukey. Zorn's aim in this paper was to study field theory and in particular to improve on the method used for obtaining results in the subject. Methods used up to that time had depended heavily on the well- ordering principle which Zermelo had proposed in 1904, namely that every set can be well-ordered. What Zorn proposed in the 1935 paper was to develop field theory from the standard axioms of set theory, together with his maximum principle rather than Zermelo's well-ordering principle.

The form in which Zorn stated his maximum principle was as follows. The principle involved chains of sets. A chain is a collection of sets with the property that for any two sets in the chain, one of the two sets is a subset of the other. Zorn defined a collection of sets to be closed if the union of every chain is in the collection. His maximum principle asserted that : *if a collection of sets is closed, then it must contain a maximal member*, that is, a set which is not a proper subset of some other in the collection. The paper then indicated how the maximum principle could be used to prove the standard field theory results.

Today we know that the *Axiom of Choice*, the *well-ordering principle*, and *Zorn's Lemma* (the name now given to Zorn's maximum principle by Tukey and now the standard name) are equivalent. Did Zorn know this when he wrote his 1935 paper? Well at the end of the 1935 paper he did say that these three are all equivalent and promised a proof in a future paper. Was Zorn's idea entirely new? Well similar maximum principles had been proposed earlier in different contexts by several mathematicians, for example Hausdorff, Kuratowski and Brouwer. Paul Campbell. Following his years at Yale, he moved to the University of California at Los Angeles where he remained until 1946. During this time Herstein was one of his doctoral students. He left the University of California to become professor at Indiana University, holding this position from 1946 until he retired in 1971. After 1947 Zorn stopped publishing mathematical papers. This does not mean that he gave up mathematics. In recent years Max became fascinated by the *Riemann Hypothesis* and possible proofs using techniques from functional analysis . He read and studied and talked about mathematics nearly every day of his life. From time to time he published a slim newsletter. He was a gentle man with a sharp wit who, during nearly half a century, inspired and charmed his colleagues at Indiana University.

Max Zorn married Alice Schlottau and they had one son Jens and one daughter Liz.

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[4]) **Zorn's Lemma** *Let $(X, \leq)$ be an inductively ordered set, i.e. every chain in X has an upper bound in X. Then X has (at least one) a maximal element.*