4.D Ordered Fields

Apart from the usual algebraic properties (e.g. addition, substraction, multiplication and division) of the real numbers, it has a natural order structure. This order structure allows us to arrange[1] the real numbers as the points on a straight line. The order structure on $\mathbb{R}$ is specified by the following axioms:

4.D.1 On $\mathbb{R}$ there is a total order $\leq$ such that for all $x, y, z \in \mathbb{R}$, we have:

(1) Monotony of addition: $x \leq y \Rightarrow x+z \leq y+z$.

(2) Monotony of multiplication: $x \leq y$ and $0 \leq z \Rightarrow xz \leq yz$

These axioms with usual arithmetic rule imply the following strict inequalities:

(1) $x < y \Rightarrow x+z < y+z$.

(2) $x < y$ and $0 < z \Rightarrow xz < yz$.

More generally, a field K with a total order $\leq$ which satisfy (1) and (2) of 4.D.1 is called an ordered field. For example, $\mathbb{Q}$ and $\mathbb{R}$ with usual order $\leq$ are

1 We use a particular direction, i.e. from left to right for the representation of real numbers as points of a straight line only for illustration and in particular, we do not use this geometrical interpretation for proofs.

ordered fields.

From the monotony laws we have the following rules for calculations with inequalities:

4.D.2 Let $(K, \leq)$ be an ordered field. For all $v, w, x, y, z \in K$, we have:

(1) $x \leq y$ and $v < w \Rightarrow x + v < y + w$.

(2) $x \leq y \Rightarrow -x \geq -y$. In particular, $x \leq 0 \Rightarrow -x \geq 0$; and $0 \leq y \Rightarrow 0 \geq -y$.

(3) $x \leq y$ and $z \leq 0 \Rightarrow xz \geq yz$. In particular, $y \geq 0$ and $z \leq 0 \Rightarrow yz \leq 0$; $x \leq 0$ and $z \leq 0 \Rightarrow xz \geq 0$.

(4) $x^2 \geq 0$ and $x^2 > 0$ iff $x \neq 0$. In particular, $1 = 1^2 > 0$.

(5) $x > 0 \Rightarrow 1/x > 0$

(6) $0 < x \leq y \Rightarrow 1/x \geq 1/y$.

Proof (1) By 4.D.1 (1), $x + v \leq y + v < y + w$.

(2) By 4.D.1(1), $-x = y + (-y - x) \geq x + (-y - x) = -y$.

(3) By (2) $-z \geq 0$ and so by 4.D.1 (2) $-xz \leq -yz$ and so $xz \geq yz$ by (2).

(4) If $x \geq 0$, then the assertion follows from 4.D.1.(2) and if $x \leq 0$, then the assertion follows from (3).

(5) $x > 0$, but if $1/x < 0$, then $1 = x(1/x) < 0$ by (3), which contradicts (4).

(6) By 4.D.1(2) $xy > 0$ and so $1/xy > 0$ by (5). Now, by 4.D.1(2) $1/x = (1/xy) \cdot y \geq (1/xy) \cdot x = 1/y$.

An element x of an ordered field K is called <u>positive</u> if $x > 0$ and is called <u>negative</u> if $x < 0$. We put

(like in the case of real numbers)

$$\operatorname{Sign} x := \begin{cases} 1, & \text{if } x>0, \\ 0, & \text{if } x=0, \\ -1, & \text{if } x<0, \end{cases}$$

and is called the <u>sign</u> or <u>signum</u> of x. The sign is multiplicative, i.e. $\operatorname{Sign}(xy) = \operatorname{Sign} x \cdot \operatorname{Sign} y$ for all $x, y \in K$. Further, from $0<1$ it follows (by induction) the positivity of all multiples $n = n \cdot 1 \in K$, $n \in \mathbb{N}^*$.

In ordered field we can also define the absolute value of elements:

<u>4.D.3</u> Let K be an ordered field. For $x \in K$,

$$|x| := \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x<0, \end{cases}$$

is called the <u>absolute value</u> or <u>modulus</u> of x

Clearly, $|x| = x \cdot \operatorname{Sign} x$. Further, for all $x, y \in K$, we have:

(1) $|x| = \operatorname{Max}(x, -x)$. (2) $|x| = |-x|$.

(3) $|x| \geq 0$ and $|x| = 0 \iff x = 0$

(4) $|xy| = |x| \cdot |y|$ and if $y \neq 0$, then $|x/y| = |x|/|y|$.

Recall that $\operatorname{Max}(x_1, \dots, x_n)$ resp. $\operatorname{Min}(x_1, \dots, x_n)$ is the greatest resp. least element among the elements $x_1, \dots, x_n \in K$, $n \in \mathbb{N}^*$.

The following triangle inequality is of special importance:

4.D.4 Triangle inequality For elements x, y in an ordered field K, we have:

$$|x+y| \leq |x|+|y| \text{ and } |x-y| \geq \big||x|-|y|\big|.$$

Proof Since $x \leq |x|$ and $y \leq |x|$, $x+y \leq |x|+|y|$, similarly, $-(x+y) \leq |x|+|y|$. Therefore $|x+y| = \operatorname{Max}(x+y, -(x+y)) \leq |x|+|y|$. This proves the first inequality and so $|x| = |(x-y)+y| \leq |x-y|+|y|$, i.e. $|x|-|y| \leq |x-y|$. Interchanging x and y, we get $|y|-|x| \leq |y-x| = |x-y|$. Together, this proves that $|x-y| \geq \operatorname{Max}(|x|-|y|, |y|-|x|) = \big||x|-|y|\big|$.

For $x, y \in K$, $d(x,y) := |y-x|$ is called the distance between x and y. We use the following notation for interval: For elements a and b in an ordered field K with $a \leq b$, we put

$$[a,b] := \{x \in K \mid a \leq x \leq b\} \quad \text{closed interval},$$
$$(a,b) := \{x \in K \mid a < x < b\} \quad \text{open interval},$$

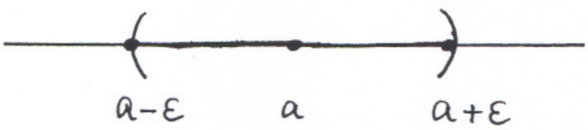
$$[a,b) := \{x \in K \mid a \leq x < b\} \quad \text{and}$$
$$(a,b] := \{x \in K \mid a < x \leq b\} \quad \text{half-open intervals}.$$

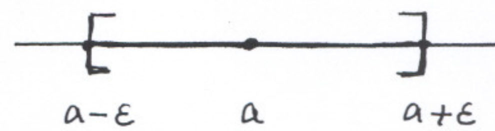
The difference $b-a$ of the end points a, b of these intervals is called the lengths of these intervals. Every interval contain all elements between any of its two elements. For $a, \varepsilon \in K$ with $\varepsilon > 0$,

$$(a-\varepsilon, a+\varepsilon) = \{x \in K \mid |x-a| < \varepsilon\}$$

$$[a-\varepsilon, a+\varepsilon] = \{x \in K \mid |x-a| \leq \varepsilon\}$$

These intervals are called the <u>open</u> respectively the <u>closed</u> ε-<u>neighbourhood</u> of a and have lengths 2ε.





A <u>neighbourhood</u> of a is simply a subset of K such that it contains an ε-neighbourhood (open or closed) of a for some $\varepsilon > 0$. This ε-neighbourhood also contains all ε'-neighbourhoods of a with $0 < \varepsilon' < \varepsilon$

It is comfortable to add two distinct elements (which are not in K) $-\infty$ and $-\infty$ to K and consider the ordered set $\overline{K} = K \cup \{\pm\infty\}$ with the greatest element ∞ and the least element $-\infty$. We therefore have:

$$-\infty \leq x \leq \infty \quad \text{for all } x \in \overline{K} = K \cup \{\pm\infty\}.$$

With this the above intervals are also defined for any two endpoints $a, b \in \overline{K}$. If either $-\infty$ or ∞ is one of the end point of an interval, then we say that this is an <u>infinite</u> or <u>unbounded</u> interval otherwise <u>finite</u> or <u>bounded</u>.

For calculations with ∞ and $-\infty$, we have the following rules:

(1) $x + \infty := \infty + x := \infty$ for all $x \in \overline{K}$, $x \neq -\infty$.

(2) $x + (-\infty) := (-\infty) + x := -\infty$ for all $x \in \overline{K}$, $x \neq \infty$.

(3) $x \cdot \infty := \infty \cdot x := \infty$, $x \cdot (-\infty) := (-\infty) \cdot x = -\infty$ for all $x \in \overline{K}$, $x > 0$.

(4) $x \cdot \infty = \infty \cdot x = -\infty$, $x \cdot (-\infty) := (-\infty) \cdot x = \infty$ for all $x \in \overline{K}$, $x < 0$

(5) $0\cdot\infty := \infty\cdot 0 := (-\infty)\cdot 0 := 0\cdot(-\infty) := 0$.

Only, the sum of $-\infty$ and ∞ resp. ∞ and $-\infty$ are not defined.

Bounded intervals are bounded sets in the sense of following definition:

<u>4.D.5 Definition</u> Let K be an ordered field. A subset A of K is said to be <u>bounded above</u> if there exists $S \in K$ such that $x \leq S$ for all $x \in A$; it is said to be <u>bounded below</u> if there exists $s \in K$ such that $x \geq s$ for all $x \in A$; it is said to be <u>bounded</u> if it is both bounded above and bounded below.

A number S (resp. s) in the above definition is called an <u>upper</u> (resp. <u>lower</u>) <u>bound</u> of A in K.

A subset $A \subseteq K$ is bounded above (resp. bounded below) iff it is a subset of an interval $(-\infty, S]$ (resp. $[s, \infty)$) with $S, s \in K$; it is bounded iff it is a subset of a finite interval $[s, S] \subseteq K$. Then it is also a subset of an interval of the form $[-R, R]$ for some $R \in K$, $R \geq 0$, i.e. $|x| \leq R$ for all $x \in A$.