4.E The concept of the convergence sequence

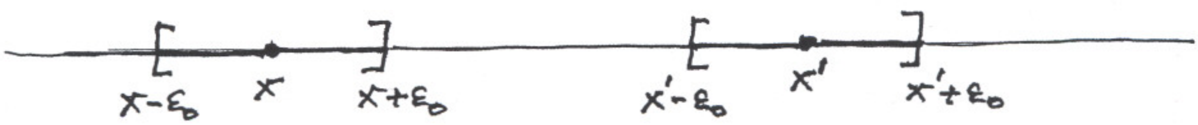
With the axioms indicated so far, i.e. $(\mathbb{R}, +, \cdot)$ is a field, ordered field, the real numbers are not yet completely characterised. For example, the rational numbers also satisfy these axioms, however $\mathbb{Q} \neq \mathbb{R}$. There are elements in $\mathbb{R}$ which are not in $\mathbb{Q}$, e.g. $\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$. For the missing <u>completeness axiom</u>, we need the concept of the <u>convergent sequence</u> in an ordered field, which we develop in this section. Throughout this section, let K denote an ordered field.

<u>**4.E.1 Definition**</u> A sequence $(x_n) = (x_n)_{n\in\mathbb{N}}$ of elements of K is said to be <u>convergent</u> (<u>in</u> K), if there exists an element $x \in K$ with the following property: for every positive $\varepsilon \in K$, there exists a $n_0 \in \mathbb{N}$ such that $|x_n - x| \leq \varepsilon$ for natural numbers $n \geq n_0$.

<u>This element x is uniquely determined by the sequence (x_n)</u>: For, suppose $x' \in K$ is another element, $x \neq x'$ satisfying the above property. Then, let $\varepsilon_0 := \frac{1}{3}|x - x'| > 0$ and let $n_0, n_0' \in \mathbb{N}$ be such that $|x_n - x| \leq \varepsilon_0$ for all $n \geq n_0$ and $|x_n - x'| \leq \varepsilon_0$ for all $n \geq n_0'$. Then for $n \geq \mathrm{Max}(n_0, n_0')$, we have

$$|x - x'| = |x - x_n + x_n - x'| \leq |x_n - x| + |x_n - x'|$$
$$\leq \varepsilon_0 + \varepsilon_0 = \tfrac{2}{3}|x - x'| < |x - x'|$$

a Contradiction.



The uniquely determined element x of the convergent sequence (x_n) is called the limiting value or the limit of the sequence (x_n) and is denoted by

$$\lim x_n = \lim_{n \to \infty} x_n$$

If x is the limit of the convergent sequence (x_n), we write it shortly as:

$$x_n \longrightarrow x \quad \text{or} \quad x_n \xrightarrow{n \to \infty} x$$

and say that the sequence (x_n) converges to x.

- A sequence (x_n) converges to x if and only if the sequence $(x_n - x)$ converges to 0. A convergent sequence with limit 0 is called a null-sequence. A sequence which is not convergent is called divergent.

A constant sequence $(x_n = x)_{n \in \mathbb{N}}$ converges to x. We say that a number x is approximated by the number y upto an error $\leq \varepsilon$ if $|y - x| \leq \varepsilon$. Therefore for a given $\varepsilon > 0$, the members after the place n_0, of a sequence converging to x, approximates the number x upto an error $\leq \varepsilon$. Naturally, for practical applications the quality of the approximation is important, therefore the question is, from which starting point n_0 the members of the sequence converging to x has absolute value from x is atmost ε. With such problems we deal later occasionally. For the concept of convergence, the speed of convergence does not play any role.

A sequence (x_n) has limiting value x if and only if every neighbourhood of x contain almost all members of the sequence, i.e. if contain all but finitely members.

4.E.2 Remark The above proof of the uniqueness of the limit value is based on the fact that in the disjoint ε_0-neighbourhoods of x resp. x' cannot simultaneously contain almost all members of the sequence (x_n).

From the definition 4.E.1 we immediately have:

4.E.3 Let (x_n) be a convergent sequence in K.

(1) Then every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is also convergent with the same limit value as the sequence (x_n), i.e. $\lim\limits_{k\to\infty} x_{n_k} = \lim\limits_{n\to\infty} x_n$.

(2) If one changes finitely many ~~members~~ of the sequence, then the sequence remain convergent with the same limit value.

Recall that $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of (x_n), if the sequence $(n_k)_{k\in\mathbb{N}}$ of indices is strictly monotone increasing, i.e. if $n_k < n_{k+1}$ for all $k\in\mathbb{N}$.

From 4.E.3 (1) it follows for example that a sequence which has not convergent subsequence or has two subsequences with different limiting values, cannot be

4.E.4 Definition Let (x_n) be a sequence in K.

(1) The sequence (x_n) is said to be <u>bounded above</u> (resp. <u>bounded below</u>) if there exists $S \in K$ (resp. $s \in K$) such that $x_n \leq S$ (resp. $x_n \geq s$) for all $n \in \mathbb{N}$.

(2) The sequence (x_n) is said to be <u>bounded</u> if it is both bounded above and bounded below.

Therefore, a sequence (x_n) is bounded above (resp. bounded below) if and only if the set $\{x_n / n \in \mathbb{N}\}$ of the members of the sequence is bounded above (resp. bounded below).

If one alters finitely many members of a sequence, its behaviour about boundedness does not change. Since every ε-neighbourhood of the limiting value of a convergent sequence contain almost all members of this sequence, we have:

<u>4.E.5</u> Every convergent sequence is bounded.

Obviously the converse <u>every bounded sequence is convergent</u> is <u>not</u> true.

4.E.6 Rules for limits Let (x_n) and (y_n) be two convergent sequences with the limiting values x and y, respectively. Then:

(1) The sum sequence $(x_n + y_n)$ is convergent and

$$\lim (x_n + y_n) = \lim x_n + \lim y_n = x + y.$$

(2) The product sequence $(x_n y_n)$ is convergent and
$$\lim (x_n y_n) = (\lim x_n)(\lim y_n) = xy.$$
In particular, $\lim(\lambda x_n) = \lambda \cdot \lim x_n = \lambda x$ for all $\lambda \in K$.

(3) If $y_n \neq 0$ for all $n \in \mathbb{N}$ and if $y \neq 0$, then the quotient sequence (x_n / y_n) is also convergent and
$$\lim\left(\frac{x_n}{y_n}\right) = \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$$
In particular, $\lim(1/y_n) = 1/\lim y_n = 1/y$.

Proof (1) Let $\varepsilon > 0$ be given. For $\varepsilon' := \varepsilon/2$ there exist $n_1, n_2 \in \mathbb{N}$ such that $|x_n - x| \leq \varepsilon'$, resp. $|y_n - y| \leq \varepsilon'$ for all $n \geq n_1$ resp. $n \geq n_2$. Then for all $n \geq n_0 := \mathrm{Max}(n_1, n_2)$ we have: $|(x_n + y_n) - (x+y)| = |(x_n - x) + (y_n - y)|$
$$\leq |x_n - x| + |y_n - y| \leq \varepsilon' + \varepsilon' = \varepsilon.$$

(2) Let $\varepsilon > 0$ be given. Then
$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$$
$$\leq |x_n y_n - x_n y| + |x_n y - xy| = |x_n||y_n - y| + |y||x_n - x|$$
By E.5 there exists $R > 0$ such that $|x_n| \leq R$ for all $n \in \mathbb{N}$. We choose for $\varepsilon' = \varepsilon / 2\,\mathrm{Max}(R, |y|)$, a natural number $n_0 \in \mathbb{N}$ with $|x_n - x| \leq \varepsilon'$ and $|y_n - y| \leq \varepsilon'$ for all $n \geq n_0$. For this $n \geq n_0$, it follows then that:
$$|x_n y_n - xy| \leq |x_n||y_n - y| + |y||x_n - x| \leq R\varepsilon' + \varepsilon'|y| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(3) In view of (2), it is enough to prove the special case given in (3). Let $\varepsilon > 0$ be given.
$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y y_n}\right| = \frac{|y_n - y|}{|y|} \cdot \frac{1}{|y_n|}$$
Since $\lim y_n = y \neq 0$, $|y|/2$-neighbourhood of 0 contains only finitely many members of the sequence (y_n).

Since $y_n \neq 0$ for all n, there exists $r>0$ such that $|y_n| \geq r$, i.e. $1/|y_n| \leq 1/r$ for all n. Now, we choose $\varepsilon' := \varepsilon \cdot r|y|$ and choose $n_0 \in \mathbb{N}$ such that $|y_n - y| \leq \varepsilon'$ for all $n \geq n_0$. Then for $n \geq n_0$, we have

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y_n - y|}{|y|} \cdot \frac{1}{|y_n|} \leq \frac{\varepsilon'}{|y| \cdot r} = \varepsilon .$$

Another useful rule is:

4.E.7 If (x_n) is a convergent sequence with limit x, then $(|x_n|)$ is convergent and $\lim |x_n| = |\lim x_n| = |x|$.

Proof Follows from $\big||x_n| - |x|\big| \leq |x_n - x|$.

For computation of limiting values one frequently use the following criterion:

4.E.8 (Sandwitch Criterion) Let (x_n), (y_n) and (z_n) be sequences. Suppose that $x_n \leq y_n \leq z_n$ for almost all $n \in \mathbb{N}$ and the sequences (x_n) and (z_n) are convergent with the same limit y. Then the sequence (y_n) is also convergent with the same limit y.

Proof Let $\varepsilon > 0$ be given. In the ε-nhd of y contain (by hypothesis) almost all members of the sequences (x_n) and (z_n) and therefore contain almost all members of the sequence (y_n).

Apart from (in the actual sense) the convergent sequences considered so far, frequently we consider (divergent) sequences which converge in the improper sense to ∞ or $-\infty$.

4.E.9 Definition A sequence (x_n) in K is said to converge (improperly) to ∞ (resp. $-\infty$) if for every $s \in K$, there exists a $n_0 \in \mathbb{N}$ such that $x_n \geq s$ (resp. $x_n \leq s$) for all $n \geq n_0$.

A sequence (x_n) converges to ∞ if and only if the sequence $(-x_n)$ converges to $-\infty$. Naturally, sequences converging to ∞ (resp. $-\infty$) are unbounded (not bounded) above (resp. below). We say that (x_n) converges absolutely to ∞ if the sequence $(|x_n|)$ of absolute values converges to ∞. In the following whenever we speak about convergent sequences, usually only sequences which are convergent in the actual sense are considered. Whenever improper convergent sequences are considered, we will usually explicitly mention this.

With the conventions (1)–(4) specified at the end of 4.D (before 4.D.5) the arithmetic rules for the calculation of the limit values also apply for the limit values $\pm\infty$. The convention $0 \cdot (\pm\infty) = 0$ does not have however any corresponding arithmetic rule for limit values. For example, in $\mathbb{R}$ $\lim(1/n) = 0$, but $\lim b_n/n$ depend substantially on the sequence (b_n), for instance for (b_n) consider the sequences $(\sqrt{n})$, (n), (n^2).