4.G Consequences of completeness

Besides the limiting values, it is appropriate to consider accumulation points of sequences:

4.G.1 Definition Let (x_n) be a sequence of real numbers. A point $x \in \mathbb{R}$ is called an accumulation point (or limit point) of (x_n) if every (however small) neighbourhood of x contain infinitely many members of the sequence (x_n).

It is clear that the limiting value of a convergent sequence is the only accumulation point of this sequence. Both 1 and -1 are accumulation points of the sequence $(-1)^n$, $n \in \mathbb{N}$. The sequence $n + (-1)^n n$, $n \in \mathbb{N}$ has the only accumulation point 0, but this sequence is not convergent. Let (x_n) be a sequence in which every rational number is a member — such a sequence exist as rational numbers are countable — the set of accumulation points of this sequence (x_n) is the whole $\mathbb{R}$.

4.G.2 Lemma Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. Then x is an accumulation point of (x_n) if and only if (x_n) has a convergent subsequence which converges to x.

Proof Suppose that (x_n) has a convergent subsequence with limiting value x. Then every nhood of x contain almost all members of this subsequence and hence contain infinitely members of the sequence (x_n).
Conversely, suppose that x is an accumulation point of (x_n). We recursively construct a subsequence (x_{n_k}) of

(x_n) which converges to x. For this, let $n_0 := 0$ and assume that $n_0, \dots, n_k \in \mathbb{N}$ are already defined. Choose $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $|x_{n_{k+1}} - x| \leq \frac{1}{k+1}$; this is possible, since $1/_{k+1}$-nhood of x contains infinitely many members of (x_n). Since the sequence $1/_k$, $k \geq 1$, is a null-sequence (see 4.F.6-(1)), the subsequence (x_{n_k}) converges to x.

The most important existence assertion about accumulation points is the following theorem:

<u>4.G.3 Theorem</u> (<u>Weierstrass-Bolzano</u>) Every bounded sequence of real numbers has an accumulation point.

<u>Proof</u> Let (x_n) be a bounded sequence. Then every member of this sequence belong to a bounded interval $[a,b] \subseteq \mathbb{R}$. For a existence of an accumulation point of (x_n), we construct a sequence $[a_n, b_n]$ $n \in \mathbb{N}$, of nested intervals, such that each interval of this sequence contain infinitely many members of the sequence (x_n). Then these nested intervals define the real number x which is clearly an accumulation point of (x_n).

We give such a sequence of nested intervals by using the well-known <u>interval halving procedure</u>. For this we put $a_0 = a$, $b_0 = b$. a_n and b_n are chosen so that in the interval $[a_n, b_n]$ (by construction) (contains) infinitely many members of the sequence (x_n). Therefore at least of the both subinterval $\left[a_n, \frac{a_n+b_n}{2}\right]$ and $\left[\frac{a_n+b_n}{2}, b_n\right]$ also contain infinitely many members of (x_n), the end points of this subinterval are defined as a_{n+1} and b_{n+1}. In case both these subintervals contain infinitely many members of (x_n), then, we take, the "left" half $\left[a_n, \frac{a_n+b_n}{2}\right]$ (on the basis of uniqueness). Since $b_n - a_n = (b-a)/_{2^n}$, we have a required sequence $[a_n, b_n]$, $n \in \mathbb{N}$ of nested intervals.

We use the Weierstrass-Bolzano theorem in the proof of Cauchy's convergence criterion.

4.G.4 Definition A sequence (x_n) of real numbers is called a Cauchy-sequence if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| \leq \varepsilon$ for all $m, n \geqslant n_0$.

It is clear that Cauchy-sequences are bounded. For, if (x_n) is a Cauchy-sequence, then for $\varepsilon = 1$, choose $n_0 \in \mathbb{N}$ with $|x_m - x_n| \leq 1$ for all $m, n \geqslant n_0$. In particular, we have $|x_m - x_{n_0}| \leq 1$ for all $m \geqslant n_0$ and hence
$|x_m| = |(x_m - x_{n_0}) + x_{n_0}| \leq |x_m - x_{n_0}| + |x_{n_0}| \leq 1 + |x_{n_0}|$.
Therefore $|x_n| \leq \mathrm{Max}(|x_0|, \cdots, |x_{n_0-1}|, 1 + |x_{n_0}|)$ for all $n \in \mathbb{N}$.

4.G.5 Lemma Every convergent sequence is a Cauchy-sequence.

Proof Let x be the limiting value of the sequence (x_n). For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x| \leq \varepsilon/2$ for all $n \geq n_0$. For all $m, n \geq n_0$, we have

$$|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x_n - x| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

4.G.6 Lemma Every Cauchy sequence with an accumulation point is convergent.

Proof Let x be an accumulation point of the Cauchy-sequence (x_n). For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| \leq \varepsilon/2$ for all $m, n \geq n_0$. Further, since x is an accumulation point of the sequence, in the $\varepsilon/2$-nhd of x there are infinitely many x_n, in particular, there exists x_{m_0} with $m_0 \geq n_0$ and $|x_{m_0} - x| \leq \varepsilon/2$.
Therefore, for all $n \geq n_0$, we have:

$$|x_n - x| = |x_n - x_{m_0} + x_{m_0} - x| \leq |x_n - x_{m_0}| + |x_{m_0} - x| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

From the theorem 4.G.3 of Weierstrass-Bolzano and the lemmas 4.G.5 and 4.5.6, we have:

4.G.7 Cauchy's convergence criterion. A sequence of real numbers is convergent if and only if it is a Cauchy-sequence.

Proof Since every Cauchy sequence is bounded, by 4.G.3 it has an accumulation point and hence is convergent by 4.G.6. The converse is proved in 4.G.5.

To prove that a Cauchy-sequence (x_n) of real numbers is convergent, one can also use the nested interval criterion 4.F.7: Let $[a_0, b_0]$, $a_0 < b_0$ be a closed interval which contain almost all members of the sequence (x_n) (use the fact that (x_n) is bounded). Recursively construct intervals $[a_{n+1}, b_{n+1}]$ which divide the interval $[a_n, b_n]$ into three (equal) parts. In (at least) one of the both extreme $1/3$-subintervals contain only finitely many members of the sequence (x_n) (Why?) and hence both the other $1/3$-subintervals together form the interval $[a_{n+1}, b_{n+1}]$ of length $2(b_n - a_n)/3$.

Now, since K is archimedean, the sequence $\left((2/3)^n\right)$ is a null-sequence and hence $\lim \frac{2}{3}(b_n - a_n) = 0$.

Therefore by 4.F.7 there exists a unique $x \in K$ with $x \in \bigcap_{n \geq 0} [a_n, b_n]$.

Now to show that $\lim x_n = x$. Let $\varepsilon \in K$, $\varepsilon > 0$ be given. Since the sequence (x_n) is a Cauchy-sequence, there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| \leq \varepsilon/2$ for all $m, n \geq n_0$. Further, since $\lim (2/3)^n = 0$, there exists $k_0 \in \mathbb{N}$ such that $b_{k_0} - a_{k_0} = (b_0 - a_0)(2/3)^{k_0} \leq \varepsilon/2$. Now, since $x_n \in [a_{k_0}, b_{k_0}]$ for infinitely many $n \in \mathbb{N}$, there exists $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and $x_{n_1} \in [a_{k_0}, b_{k_0}]$ and hence $|x_{n_1} - x| \leq b_{k_0} - a_{k_0} \leq \varepsilon/2$, because $x \in [a_{k_0}, b_{k_0}]$. Therefore for all $n \geq n_0$, we have $|x_n - x| = |x_n - x_{n_1} + x_{n_1} - x| \leq |x_n - x_{n_1}| + |x_{n_1} - x| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. This proves (that) $\lim x_n = x$.

At the end of this paragraph we further study some concepts for subsets of real numbers.

4.G.8 Definition Let $A \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ which is bounded above (resp. bounded below). A least upper (resp. greatest lower) bound of A is called the upper limit or the supremum (resp. the lower limit or the infimum) of A and is usually denoted by $\operatorname{Sup} A$ (resp. $\operatorname{Inf} A$).

If upper and lower limits exists, then clearly they are uniquely determined.

An element $S \in \mathbb{R}$ is the upper limit of $A \subseteq \mathbb{R}$ if and only if: (1) S is an upper bound of A (2) For every upper bound S' of A, we have $S \leq S'$. The condition (2) is equivalent to the following: (2′) For every $\varepsilon > 0$, there exists $x \in A$ with $S - \varepsilon < x$.

The corresponding assertion hold for the lower limit. For $a, b \in \mathbb{R}$, $a < b$, the upper and lower limit of the intervals $[a, b]$, $]a, b[$, $[a, b[$ and $]a, b]$ are a and b respectively. The subset $\{1/n \mid n \in \mathbb{N}^*\}$ of unit-fractions has upper limit 1 and lower limit 0. The following theorem guarantees the existence of upper and lower limits.

4.G.9 Theorem on upper and lower limits. Every non-empty bounded above (resp. bounded below) subset of real numbers has an upper limit (resp. lower limit) in $\mathbb{R}$.

Proof Let S be an upper bound of a bounded above non-empty subset A of $\mathbb{R}$ and let $a \in A$. For the construction of the upper limit of A, we again use the process of nested intervals. We construct intervals $[a_n, b_n]$ such that at least one element of A is contained in $[a_n, b_n]$ and b_n is an upper bound for A. Then clearly these nested intervals define a unique real number which is the upper limit of A.

Put $a_0 = a \in A$ and $b_0 = S$. Assuming that a_n and b_n are already defined with required properties, let

$$a_{n+1} = \begin{cases} a_n, & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound for } A, \\ \frac{a_n+b_n}{2}, & \text{otherwise,} \end{cases}$$

$$b_{n+1} = \begin{cases} \frac{a_n+b_n}{2}, & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound for } A, \\ b_n, & \text{otherwise.} \end{cases}$$

Therefore $[a_{n+1}, b_{n+1}]$ is the "left" resp. "right" half of $[a_n, b_n]$ according as the midpoint of $[a_n, b_n]$ is an upper bound of A or not an upper bound of A.

The case of the lower limit can be proved analogously or use the existence of the upper limit for the subset $-A := \{-x \mid x \in A\}$.

It is comfortable to define the supremum (resp. the infimum) of a (non-empty) subset which is not bounded above (resp. not bounded below) A of $\mathbb{R}$ to be ∞ (resp. $-\infty$). With this convention: <u>for every subset A of $\mathbb{R}$ has a supremum and infimum in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$</u>.

For the empty set $\emptyset$, $-\infty$ is the supremum and ∞ is the infimum.

4.G.10 Definition Let A be a subset of $\mathbb{R}$ and $x \in \mathbb{R}$.

(1) The point x is called a closure point of A, if every neighbourhood of A contains at least one element of A.

(2) The point x is called an accumulation point of A, if every neighbourhood of A contains an infinitely many elements of A.

(3) The point x is called an inner point or an interior point of A, if A is a neighbourhood of x.

(4) The set A is called closed if every boundary point of A belongs to A.

(5) The set A is called open if every point of A is an interior point of A.

Clearly a point $x \in \mathbb{R}$ is an accumulation point of A if in every nhd of x contains a point of A which is different from x. The set of closure points of A in $\mathbb{R}$ is denoted by $\overline{A}$ and the set of interior points of A is denoted by $\overset{\circ}{A}$. A point $x \in \mathbb{R}$ is a closure point of A if and only if there exists a sequence x_n, $n \in \mathbb{N}$ in A such that $x = \lim x_n$; x is an accumulation point of A if and only if members of the sequence x_n, $n \in \mathbb{N}$ can be chosen different from x.

From the above definitions we have:

4.G.11 Theorem For every subset A of $\mathbb{R}$, we have:

(1) Every accumulation point of A is a closure point of A and every closure point of A which is not in A, is a closure point of A.

(2) $\mathring{A} \subseteq A \subseteq \bar{A}$. Further $A = \bar{A}$ if and only if A is closed and $\mathring{A} = A$ if and only if A is open.

(3) A is open (resp. closed) in $\mathbb{R}$ if and only if the complement $\mathbb{R} \setminus A$ of A in $\mathbb{R}$ is closed (resp. open) in $\mathbb{R}$.

Examples of closed (resp. open) subsets of $\mathbb{R}$ are the closed (resp. open) intervals. The half-open intervals $]a,b]$ and $[a,b[$, $a, b \in \mathbb{R}$, $a < b$ are neither open nor closed.

4.G.12 Remark The topological concepts introduced in 4.G.10 here always refer to the whole $\mathbb{R}$ (resp. on the whole $\mathbb{C}$ in the next paragraph). With an appropriate concepts for any topological spaces, in particular, for an arbitrary subsets of $\mathbb{R}$ (resp. of $\mathbb{C}$) will be dealt in later sections (on topological spaces).

Clearly the supremum (resp. infimum) of a bounded above (resp. bounded below) subset of $\mathbb{R}$ is a closure point of this subset. Therefore we have:

4.G.13 Theorem Every non-empty bounded closed subset in $\mathbb{R}$ contains its infimum and its supremum, i.e. a least and a greatest element.

4.G.14 Lemma The set of the accumulation points of a sequence of real numbers is closed.

Proof Let x be a closure point of the set of the accumulation points of the sequence (x_n). In every open ε-nhd of x contain an accumulation point y of (x_n). Since this ε-nhd is also a nhd of y, it contain an infinitely many members of the sequence (x_n). Therefore x is an accumulation point of (x_n).

Let (x_n) be a bounded sequence of real numbers. By the theorem 4.G.3 of Weierstrass-Bolzano, the set of its accumulation points is non-empty and naturally bounded. Further, by 4-G.14 it is closed and hence by 4.G.13 it contains a least and a greatest elements. These accumulation points (which coincide in case of convergent sequences) are called the limit inferior and the limit superior of the sequence and are denoted by $\liminf x_n$ and $\limsup x_n$ resp. Moreover, the accumulation point constructed in the proof of the theorem of Weierstrass-Bolzano is the limit inferior.

For an arbitrary sequence (x_n) of real numbers, $-\infty$ (resp. ∞) is an accumulation point of (x_n) if there-exists a subsequence of (x_n) which converges to $-\infty$ (resp. ∞). Therefore every sequence of real numbers has a least and a greatest accumulation point in $\overline{\mathbb{R}} =$

$= \mathbb{R} \cup \{-\infty, \infty\}$, i.e. a limit inferior and limit superior.

For example ∞ is the limit inferior and the limit superior of the sequence n, $n \in \mathbb{N}$, of the natural numbers.

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