# MA-302 Advanced Calculus

## 2. Surface area and Length

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**2.1. a).** (Plane Polar coordinates) Let $t \mapsto r(t)$ and $t \mapsto \varphi(t)$ be continuously differentiable functions on the interval $[a, b]$. The curve

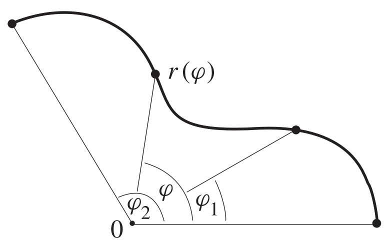
$$t \mapsto r(t)\,\big(\cos\varphi(t)\,,\,\sin\varphi(t)\big)$$

in $\mathbb{R}^2$, with the Euclidean standard norm, has the length

$$L_a^b = \int_a^b (\dot{r}^2 + r^2\dot{\varphi}^2)^{1/2}\,dt\,.$$

In particular, if $\varphi(t) = t$ for all $t$, then $r(t) = r(\varphi)$, and the length of the curve between the angles $\varphi_1$ and $\varphi_2$ is equal to

$$L_{\varphi_1}^{\varphi_2} = \int_{\varphi_1}^{\varphi_2} \Big(\frac{d^2r}{d\varphi^2} + r^2\Big)^{1/2}\,d\varphi\,.$$




**b).** (Space Polar coordinates) Let $t \mapsto r(t)$, $t \mapsto \varphi(t)$ and $t \mapsto \lambda(t)$ be continuously differentiable functions on the interval $[a, b]$. The curve

$$t \mapsto r(t)\,\big(\cos\varphi(t)\cos\lambda(t)\,,\,\cos\varphi(t)\sin\lambda(t)\,,\,\sin\varphi(t)\big)$$

in $\mathbb{R}^3$, with Euclidean standard norm, has the length

$$L_a^b = \int_a^b (\dot{r}^2 + r^2\dot{\varphi}^2 + r^2\dot{\lambda}^2\cos^2\varphi)^{1/2}\,dt\,.$$

**c).** The graph $t \mapsto \big(t, g_1(t), \ldots, g_n(t)\big)$ of a continuously differentiable curve $g : [a, b] \to \mathbb{R}^n$ has (with respect to the Euclidean standard norm) the length

$$L_a^b = \int_a^b (1 + \dot{g}_1^2 + \cdots + \dot{g}_n^2)^{1/2}\,dt\,.$$

**d).** Compute the length of the perimeter of the unit circle in $\mathbb{R}^2$ with respect to the maximum norm and with respect to the sum-norm of $\mathbb{R}^2$.

**e).** Let $c \in \mathbb{R}^\times$ and let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the logarithic spiral $\gamma(t) := (e^{ct}\cos t, e^{ct}\sin t)$. For $[a, b] \subseteq \mathbb{R}$, compute the arc-length $L_{a,b} := L_a^b(\gamma|[a, b])$. Does the limit $\lim\limits_{a\to-\infty} L_{a,0}$ exists?

**2.2.** Compute the arc-length of the ellipse $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (a\cos t, b\sin t)$ with the help of the complete elliptic integral [1])

**2.3.** Let $V$ be a finite dimensional normed $\mathbb{R}$-vector space and let $f : [a, b] \to V$ be a rectifiable curve of the length $L$ in $V$.

**a).** Show that the arc-length $t \mapsto s(t) := L_a^t(f)$ is a monotone increasing and surjective continuous function $[a, b] \to [0, L]$ and there exists a unique rectifiable curve $g : [0, L] \to V$ such that $f = g \circ s$, and $L_0^{s_0}(g) = s_0$ for every point $s_0 \in [0, L]$. Further, the function $s$ is strictly monotone increasing if and only if $f$ is non-constant on every subinterval of $[a, b]$ with more than one point. In this case $s$ defines a parametrisation of $f$, and $g = f \circ s^{-1}$ is the arc-length parametrised curve corresponding to $f$.

**b).** Let $\varphi : [\alpha, \beta] \to [a, b]$ be a monotone and surjective continuous function. Then the curve $f \circ \varphi$ ia also rectifiable and $L_\alpha^\beta(f \circ \varphi) = L = L_a^b(f)$.

**c).** Show that the continuous curve $g : [0, 1] \to \mathbb{R}^2$ mit $g(t) := \big(t, t\cos(1/t)\big)$ for $t \neq 0$ and $g(0) := (0, 0)$ and the differentiablle curve $h : [0, 1] \to \mathbb{R}^2$ with $h(t) := \big(t, t^2\cos(1/t^2)\big)$ für $t \neq 0$ and $h(0) := (0, 0)$ are not rectifiable.

**2.4.** Let $V$ be a finite dimensional normed $\mathbb{R}$-vector space with basis $v_i$, $i \in I$.

**a).** Show that a curve $f : [a, b] \to V$ with $t \mapsto \sum_{i\in I} f_i(t)\, v_i$ is rectifiable if and only if all the component functions $f_i : [a, b] \to \mathbb{R}$ are rectifiable.

(The length of a curve in $\mathbb{R}$ is also called its variation. Therefore the concept of the Variation is defined for arbitrary function $h : [a, b] \to \mathbb{R}$ as the supremum over all sums $\sum_{j=0}^{m-1} |h(t_{j+1}) - h(t_j)|$, where $t_0, \ldots, t_m$ runs over all finite sequences with $a \le t_0 \le \cdots \le t_m \le b$.)

**b).** Show that if $h : [a, b] \to \mathbb{R}$ has a finite variation, then $h = f - g$ for some monotone increasing functions $f, g : [a, b] \to \mathbb{R}$, moreover, if $h$ is continuous then one can choose both $f$ and $g$ continuous functions. ( **Hint:** One can define $f(t)$ for $t \in [a, b]$ as the supremum over all sums $\sum_{j=0}^{m-1} \mathrm{Max}\,\big(h(t_{j+1}) - h(t_j), 0\big)$, $a \le t_0 \le \cdots \le t_m \le t$.)

**2.5.** Let $\gamma : I \to \mathbb{C}$ be a closed continuous curve. Show that

**a).** The function $\mathrm{W}(\gamma, -) : \mathbb{C} \setminus \mathrm{im}(\gamma) \to \mathbb{Z}$, $z \mapsto \mathrm{W}(\gamma, z)$ is locally constant.

**b).** The sets $\mathrm{Int}\,\gamma := \{z \in \mathbb{C} \setminus \mathrm{im}(\gamma) \mid \mathrm{W}(\gamma, z) \neq 0\}$ and $\mathrm{Ext}\,\gamma := \{z \in \mathbb{C} \setminus \mathrm{im}(\gamma) \mid \mathrm{W}(\gamma, z) = 0\}$ are open in $\mathbb{C}$. (These sets are called the inside or interior and the outside or exterior of $\gamma$, respectively. In particular, $C = \mathrm{Int}\,\gamma \cup \mathrm{im}\gamma \cup \mathrm{Ext}\,\gamma$ is a disjoint decomposition of $\mathbb{C}$. ).

**c).** The set $\mathrm{Int}\,\gamma$ is bounded and the set $\mathrm{Ext}\,\gamma$ is never empty and always unbounded. More precisely, if $\mathrm{im}(\gamma) \subset \mathrm{B}(a; r) := \{z \in \mathbb{C} \mid |z - a| < r\}$, then $\mathrm{Int}\,\gamma \subset \mathrm{B}(a; r)$ and $\mathbb{C} \setminus \mathrm{B}(a; r) \subset \mathrm{Ext}\,\gamma$

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[1]) For every $k \in [0, 1]$, the improper integral $E(k) := \int_0^1 \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}}\, dt$ exists and is equal to the integral $\int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\tau}\, d\tau$ and is called the complete elliptic integral.