MA-302 Advanced Calculus

4. Total Differentials, Chain Rule

Gottfried Leibniz (1646-1716) †

The exercise 4.2 is only to get practise and its solution need not be submitted.

4.1. a). Compute the partial derivatives and the total differential of the determinante function $\operatorname{Det}: \mathrm{M}_n(\mathbb{K}) \to \mathbb{K}$. At which points $\mathfrak{A} \in \mathrm{M}_n(\mathbb{K})$ the total differential is identically 0? what is the linear form $\operatorname{DDet}(\mathfrak{E}_n)$?

b). Let V be a Euclidean vector space. Compute the total differential and the Jacobi-determinant of the map $x \mapsto x/\|x\|^2$, $x \in V \setminus \{0\}$.

4.2. Let V be an Euclidean vector space of dimension n with the inner product $\langle -, - \rangle$. Compute the gradient of the following functions.

a). $x \mapsto \langle x, x \rangle$, $x \in V$. $\quad x \mapsto \|x\|$, $x \in V \setminus \{0\}$.

b). $x \mapsto \Phi(x, x)$, $x \in V$, where Φ is a symmetric bilinear form on V (with corresponding self adjoint operator $f = f_\Phi$).

c). $x \mapsto f(x)\, g(x)$, $x \in G$, where $f, g : G \to \mathbb{R}$ are differentiable with gradients grad f, grad g.

d). $x \mapsto f(x)/g(x)$, $x \in G$, where $f, g : G \to \mathbb{R}$ differentiable with gradients grad f, grad g and g has no zero in G.

e). $x \mapsto (H \circ f)(x)$, $x \in G$, where $f : G \to \mathbb{R}$ differentiable with the gradient grad f and $H : I \to \mathbb{R}$ (with $I \supseteq f(G)$) differentiable.

f). $x \mapsto R_\Phi(x) = \Phi(x, x)/\langle x, x \rangle$, $x \in V - \{0\}$, where Φ has the same meaning as in c). (R_Φ is called the Rayleigh-Quotient corresponding to Φ.)

g). $x \mapsto \|x\|^\alpha$, $x \in V \setminus \{0\}$, $\alpha \in \mathbb{R}$.

h). $x \mapsto H(\|x\|)$, $x \in V \setminus \{0\}$, where $H : \mathbb{R}_+^\times \to \mathbb{R}$ is differentiable.

i). $(x_1, \ldots, x_n) \mapsto \Delta(x_1, \ldots, x_n)$ on $V^n = V \oplus \cdots \oplus V$, where V is oriented with the cannonical determinant function Δ.

4.3. Let V, W be Euclidean vector spaces and let $f, g : G \to W$ be maps on an open subset $G \subseteq V$, $h : H \to \mathbb{R}$ be a function on an open subset $H \supseteq f(G)$. Assume that f, g are differentiable at $x_0 \in G$ and h is differentiable at $f(x_0)$. Then show that:

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† **Gottfried Leibniz** was the son of Friedrich Leibniz, a professor of moral philosophy at Leipzig. Leibniz's mother was Catharina Schmuck, the daughter of a lawyer and Friedrich Leibniz's third wife.

At the age of seven, Leibniz entered the Nicolai School in Leipzig. In 1661, at the age of fourteen, Leibniz entered the University of Leipzig. It may sound today as if this were a truly exceptionally early age for anyone to enter university, but it is fair to say that by the standards of the time he was quite young but there would be others of a similar age. He studied philosophy, which was well taught at the University of Leipzig, and mathematics which was very poorly taught.

a). (Chain rule for gradients). $(\operatorname{grad} h \circ f)(x_0) = (\widehat{\mathrm{D}}f)_{x_0}\big((\operatorname{grad} h)(f(x_0))\big)$, where $(\widehat{\mathrm{D}}f)_{x_0} : W \to V$ is the adjoint of the linear map $(\mathrm{D}f)_{x_0} : V \to W$.

b). $\big(\operatorname{grad}\langle f, g\rangle\big)(x_0) = (\widehat{\mathrm{D}}f)_{x_0}\big(g(x_0)\big) + (\widehat{\mathrm{D}}g)_{x_0}\big(f(x_0)\big)$.

4.4. Let V be a finite dimensional real vector space and let $f : V \setminus \{0\} \to \mathbb{R}$ be a differentiable function.

a). If f is homogeneous of degree $\alpha \in \mathbb{R}$, then every directional derivative $\mathrm{D}_v f$ of f is homogeneous of degree $\alpha - 1$. (A function $f : V \setminus \{0\} \to \mathbb{R}$ is called homogeneous of degree $\alpha \in \mathbb{R}$ if $f(tx) = r^\alpha f(x)$ for every $x \in V \setminus \{0\}$.)

b). Show that f is homogeneous of degree $\alpha \in \mathbb{R}$ if and only if for all $v \in V \setminus \{0\}$ we have

$$(\mathrm{D}_v f)(v) = \alpha f(v) \quad \text{(Euler's Equation)}.$$

In particular, if f is homogeneous of degree α and V has an inner product, then $\langle v, \operatorname{grad} f(v)\rangle = \alpha f(v)$,

4.5. a). Let $G \subseteq V$ be an open connected subset and let $f : G \to W$ be a differentiable map. If the total differential $\mathrm{D}f : G \to \operatorname{Hom}_{\mathbb{K}}(V, W)$ of f is the constant map and is equal to $T \in \operatorname{Hom}_{\mathbb{K}}(V, W)$. Then show that f is the affine map $x \mapsto y_0 + T(x)$ with fixed $y_0 \in W$.

b). Compute the total differential of the function $f(x, y) := \arctan(x/y) + \arctan(y/x)$ on $\mathbb{R}_+^\times \times \mathbb{R}_+^\times$.

In 1672 Leibniz went to Paris, his first object in Paris was to make contact with the French government but, while waiting for such an opportunity, Leibniz made contact with mathematicians and philosophers there. In Paris Leibniz studied mathematics and physics under Christiaan Huygens.

In January 1673, Leibniz visited the Royal Society, and talked with Hooke, Boyle and Pell. While explaining his results on series to Pell, he was told that these were to be found in a book by Mouton. The next day he consulted Mouton's book and found that Pell was correct. Leibniz realised that his knowledge of mathematics was less than he would have liked so he redoubled his efforts on the subject and also began to study the geometry of infinitesimals. The Royal Society of London elected Leibniz a fellow on 19 April 1673.

In August 1675 Tschirnhaus arrived in Paris and he formed a close friendship with Leibniz which proved very mathematically profitable to both.

It was during this period in Paris that Leibniz developed the basic features of his version of the calculus. In 1673 he was still struggling to develop a good notation for his calculus and his first calculations were clumsy. On 21 November 1675 he wrote a manuscript using the $\int f(x)\mathrm{d}x$ notation for the first time. In the same manuscript the product rule for differentiation is given. By autumn 1676 Leibniz discovered the familiar $\mathrm{d}(x^n) = nx^{n-1}\mathrm{d}x$ for both integral and fractional n.

Newton wrote a letter to Leibniz, which took some time to reach him. The letter listed many of Newton's results but it did not describe his methods. Leibniz replied immediately but Newton , not realising that his letter had taken a long time to reach Leibniz, thought he had had six weeks to work on his reply. Certainly one of the consequences of Newton's letter was that Leibniz realised he must quickly publish a fuller account of his own methods.

Newton wrote a second letter to Leibniz on 24 October 1676 which did not reach Leibniz until June 1677 by which time Leibniz was in Hanover. This second letter, although polite in tone, was clearly written by Newton believing that Leibniz had stolen his methods. In his reply Leibniz gave some details of the principles of his differential calculus including the rule for differentiating a function of a function.