MA-302 Advanced Calculus

5. Taylor's formula, Local extrema, Laplace operator

Brook Taylor[†]
(1685-1731)

Pierre-Simon Laplace[††]
(1749-1827)

Otto Hesse[†††]
(1811-1874)

The exercises 5.1 and 5.2 are only to get practise and its solution need not be submitted.

5.1. For the following functions f compute the Taylor's polynomial of degree $< k$ at the point x_0.

a). $f(x, y) = x^2 + 4xy^3 + y + 3y^2$ for $x_0 = (0, 0)$ resp. $x_0 = (1, 2)$ and $k = 3, 4, 5$.

b). $f(x, y) = (x - y)/(x + y)$ for $x_0 = (1, 1)$ and $k = 4$.

c). $f(x, y, z) = e^{x+y+z}/(e^x + e^y)(e^y + e^z)(e^x + e^z)$ for $x_0 = (0, 0, 0)$ and $k = 3$.

5.2. Find the critical points of the following real functions f and determine their local extrema.

a). $f : V \to \mathbb{R}$ with $f(x) := \Phi(x, x)$, where Φ is a real symmetric bilinear form on V of type (p, q).

b). $f(x, y) = x + xy + y^2$.

c). $f(x, y) = \cos x \, \cos y \, \cos(x + y)$.

***d).** $f(x, y) = \dfrac{x}{y+z} + \dfrac{y}{x+z} + \dfrac{z}{x+y}$ on $(\mathbb{R}_+^\times)^3$.

e). $f(x, y, z) = g(x, y, z) \exp\big(-(x^2 + y^2 + z^2)\big)$, where g is a linear form on $\mathbb{R}^3$.

f). (Exercise of Huygens)

$$f(x_1, \ldots, x_n) := \frac{x_1 \cdots x_n}{(a + x_1)(x_1 + x_2) \cdots (x_{n-1} + x_n)(x_n + b)}$$

on $(a, b)^n := \underbrace{(a, b) \times \cdots \times (a, b)}_{n-\text{times}}$, where a and b are positive real numbers with $a < b$.

g). $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 - 2x_1 \cdots x_n$.

5.3. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y) = (x^2 + y^2)(x^2 - 3x + y^2) + 2x^2$ has a local minimum at 0 on every line passing through 0, but f has no local minimum at 0.

5.4. (Principle of least squares) Let $v_1, \ldots, v_n$ be vectors in an Euclidean or unitary vector space V and $a_1, \ldots, a_n \geq 0$ be weights with $a_1 + \cdots + a_n > 0$. Show that the function $f(x) = \sum_{j=1}^n a_j \, \|x - v_j\|^2$ of weighted sum of square distances has a unique local minimum in V namely at the centre of gravity of the points $v_1, \ldots, v_n$ with respect to the weights $a_1, \ldots, a_n$.

5.5. Let $\alpha_1, \ldots, \alpha_n \geq 0$. Show that the function $(x_1, \ldots, x_n) \mapsto x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is concave on $(\mathbb{R}_+^\times)^n$ if and only if $\alpha_1 + \cdots + \alpha_n \leq 1$. Further, it is strict-concave if and only if $\alpha_1 + \cdots + \alpha_n < 1$ and all α_i are positive.

5.6. Let V be Euclidean and let $f, g : G \to \mathbb{R}$ be two-times continuously differentiable functions. For the Laplace-Operator Δ prove the Product rule :

$$\Delta(fg) = f \, \Delta g + 2 \langle \operatorname{grad} f \, , \, \operatorname{grad} g \rangle + g \, \Delta f \, .$$

In particular, the product of two harmonic functions f, g is harmonic if and only if their gradients are orthogonal at every point.

5.7. Let V be an n-dimensional Euclidean vector space, $n \geq 1$.

a). If $f : \mathbb{R}_+^\times \to \mathbb{R}$ is two-times continuously differentiable function, then show that the function $x \mapsto f\big(\|x\|\big)$ is also two-times continuously differeniable on $V \setminus \{0\}$ and we have

$$\Delta\big(f\|x\|)\big) = f''\big(\|x\|\big) + (n-1)\,\frac{f'\big(\|x\|\big)}{\|x\|}\,.$$

b). The function $x \mapsto 1/\|x\|^{n-2}$ auf $V \setminus \{0\}$ is harmonic. In the case $n = 2$, the function $x \mapsto \ln\|x\|$ is also harmonic.

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† **Brook Taylor (1665-1731)** was born on 18 Aug 1685 in Edmonton, Middlesex, England and died on 29 Dec 1731 in Somerset House, London, England. In addition to his Mathematics, Taylor worked on Kepler's second law of planetary motion, capillary action, magnetism, thermometers, vibrating strings; he also discovered the basic principles of perspective. His life, however, suffered a series of personal tragedies beginning around 1721, he married Miss Brydges from Wallington in Surrey. Although she was from a good family, it was not a family with money and Taylor's father strongly objected to the marriage. The result was that relations between Taylor and his father broke down and there was no contact between father and son until 1723. It was in that year that Taylor's wife died in childbirth. The child, which would have been their first, also died. After the tragedy of losing his wife and child, Taylor returned to live with his father and relations between the two were repaired. Two years later, in 1725, Taylor married again to Sabetta Sawbridge from Olantigh in Kent. This marriage had the approval of Taylor's father who died four years later on 4 April 1729. Taylor inherited his father's estate of Bifons but further tragedy was to strike when his second wife Sabetta died in childbirth in the following year. On this occasion the child, a daughter Elizabeth, did survive.

†† **Pierre-Simon Laplace (1749-1827)** was born on 23 March 1749 in Beaumont-en-Auge, Normandy, France and died on 5 March 1827 in Paris, France. Laplace was a Mathematician and theoretical astronomer who was so famous in his own time that he was known as the Newton of France. His main interests throughout his life were celestial mechanices, the theory of probability, and personal advancement.

It does appear that Laplace was not modest about his abilities and achievements, and he probably failed to recognise the effect of his attitude on his colleagues. Laplace had always changed his views with the changing political events of the time, modifying his opinions to fit in with the frequent political changes which were typical of this period. This way of behaving added to his success in the 1790s and 1800s but certainly did nothing for his personal relations with his colleagues who saw his changes of views as merely attempts to win favour. To balance his faults, Laplace was always generous in giving assistance and encouragement to younger scientists. From time to time he helped forward in their careers such men as the Gay-Lussac, the traveler and naturalist Humbolt, the physicist Poisson, and appropriately the young Cauchy, who was destined to become one of the chief architects of nineteeth century mathematics.

††† **Otto Hesse (1811-1874)** was born on 22 April 1811 in Königsberg, Germany (now Kaliningrad, Russia), died on 4 Aug 1874 in Munich, Germany. and studied under Jacobi at Königsberg. Hesse's main work was in the development of the theory algebraic functions and the theory of invariants. He introduced the Hessian determinant in 1842 during an investigation of cubic and quadratic curves. His work was influenced by Steiner, particularly work he did on the geometrical interpretation of algebraic transformations. Hesse worked on some topics that Cayley was also working on and both produced a theory of homogeneous forms which they published at the same time.