MA-302 Advanced Calculus

7. Implicit function theorem

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(1863-1942)

René Maurice Fréchet ††
(1878-1973)

The exercises 7.1 and 7.2 are only to get practice, their solutions need not be submitted.

7.1. For the following functions determine the critical points. At the other points (i.e. non- critical points) x_0 find the local differentiable solutions of the equation $F(x) = F(x_0)$ explicitly. Further, for these solutions find the domain and total differentials.

a). $F(x_1, \ldots, x_n) = x_1 \cdots x_n - c$ on $\mathbb{K}^n$ ($c \in \mathbb{K}$ constant).

b). $F(x_1, \ldots, x_n) = x_1^m + \cdots + x_n^m - c$ on $\mathbb{K}^n$ ($c \in \mathbb{K}$ constant).

c). $F(x_1, \ldots, x_n) = x_1 + \cdots + x_n - x_1 \cdots x_n$ on $\mathbb{K}^n$.

d). $F(x_1, \ldots, x_n) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$ on $\mathbb{R}^n$, $n \geq p+q$.

e). $F(x, y, z) = xy + yz + zx$ on $\mathbb{R}^3$.

7.2. Determine whether or not the following system of equations have solutions with respect to the given variables at the given points and find the Taylor-expansion upto order 2 for the solutions which exist.

a). $F(x, y, z) = x \cos y + y \cos z + z \cos x = 1$ at $(0, 0, 1)$ with respect to z.

b). $F(x, y, z) = z^3 - z(2y - x) + x + y = 2$ at $(1, 1, 1)$ with respect to z.

c). $F(x, y, u, v) = (xu + yv\,, uv - xy) = (0, 5)$ at $(1, -1, 2, 2)$ with respect to (u, v).

d). $F(x, y, u, v) = (2x - u^2 + v^2, y - uv) = (0, 0)$ at $(0, 1, 1, 1)$ with respect to (u, v).

e). $F(x, y, z) = (x - z(x + y - 4), y + z \ln(2 - x)) = (1, 2)$ at $(1, 2, 0)$ with respect to (x, y).

7.3. Let $k \geq 3$ and let $f : U \to \mathbb{K}$ be a k-times continuously differentiable function on the neighbourhood U of $0 \in \mathbb{K}$. The equation $F(x, y, z) := z - y - x f(z) = 0$ is implicitly defined by a k-times continuously differentiable function $z = g(x, y)$ in an open neighbourhood of $(0, 0) \in \mathbb{K}^2$.

a). Show that (by induction on j)

$$\frac{\partial^{j+1} g}{\partial x^{j+1}}(x, y) = \frac{\partial^j}{\partial y^j}\Big(f^{j+1}\big(g(x, y)\big)\,\frac{\partial g}{\partial y}(x, y)\Big)\,, \quad j = 0, \ldots, k-1\,.$$

b). For a fixed y prove that the Taylor-polynomial of degree k of g in x is:

$$g(x, y) = y + xf(y) + \frac{x^2}{2!}\frac{d}{dy}\big(f^2(y)\big) + \cdots + \frac{x^k}{k!}\frac{d^k}{dy^k}\big(f^k(y)\big)\,.$$

c). For a fixed $y \in U$, the local solution $z = g(x, y)$ can be obtained by the equation $z - xf(z) = y$ at the point $x = 0$, $z = y$ and the recursion relations

$$g_0(x, y) = y\,, \quad g_{m+1}(x, y) = y + xf\big(g_m(x, y)\big)\,, \quad m > 0\,.$$

7.4. Let $f : U \to \mathbb{K}$ be a k-times continuously differentiable function on the open neighbourhood U of $0 \in \mathbb{K}$ with $f(0) \neq 0$. Then the equation $x = z/f(z)$ defines a k-times continuously differentiable function $z = g(x)$ in an open neighbourhood of $0 \in \mathbb{K}$ and has the Taylor-expansion

$$g(x) = \sum_{m=1}^{k} \frac{a_m}{m!} x^m + R \qquad \text{with} \qquad a_m = (f^m)^{(m-1)}(0)\,, \quad m = 1, \dots, k\,.$$

Moreover, if f is analytic, then so is g. (**Hint:** Use the exercise 7.3)

7.5. Using the exercise 7.3 find the expansion of the solution $z = g(x, y) = \sum_{m=0}^{\infty} a_m(y)\, x^m$ of

$$z - y - xz^n = 0\,,\ n \geq 2\,, \quad \text{respectively} \quad z - y - x \sin z = 0 \qquad \text{(Kepler's Equation)}$$

in an open neighbourhood of 0 (**Hint:** for the Kepler's equation: prove that the Newton's-method $g_0(x, y) = y\,,\quad g_{m+1}(x, y) = y + x \sin\big(g_m(x, y)\big)\,,\ m \geq 0\,,$ is convergent for every $x \in \mathbb{R}$ with $|x| < 1$ and for all $y \in \mathbb{R}$ to the solution $z = g(x, y)$ of the Kepler's equation.)

† **William Henry Young (1863-1942)** was born on 20 Oct 1863 in London, England and died on 7 July 1942 in Lausanne, Switzerland William Young's father was Henry Young, a grocer, and his mother was Hephzibah Jeal. William was his parents' eldest son He attended the City of London School where the headmaster was particularly fascinated by mathematics. He immediately saw the potential that Young had for mathematics and he encouraged him in that direction. In 1881 Young entered Peterhouse, Cambridge to begin his undergraduate studies of mathematics.

At Cambridge Young was an outstanding student showing far more mathematical ability than any of the other students in his year. However to achieve the position of First Wrangler (the top position in the list of First Class graduates) in the Mathematical Tripos required enormous dedication and training in the type of examination questions set in the Tripos. It would be fair to say that the First Wrangler was the most skilled at answering Tripos questions rather than the best mathematician and many of the great mathematicians who attended Cambridge failed to gain this distinction. Young was one such student for he made a very conscience decision that becoming First Wrangler was less important to him than having varied interests, both academic and sporting, at university. He was fourth wrangler in 1884. While at Cambridge he put aside the Baptist religion of his family and as baptised into the Church of England.

Although many famous mathematicians who attended Cambridge failed to become First Wrangler, many of those who failed did become Smith's Prizemen. However Young did not even submit an essay for this prize but submitted an essay for a theology prize instead. He won the theology prize and he decided to remain at Cambridge earning money by privately coaching students for the mathematical tripos. He did not undertake any mathematical research although he was a Fellow of Peterhouse between 1886 and 1892. One of the students Young tutored was GRACE EMILY CHISHOLM, who studied mathematics at Girton College. She then went to Göttingen where she was supervised for her doctorate by KLEIN, returning in 1885 after the award of the doctorate. It is extremely unlikely that Young would have ever become interested in research had it not been that he married Grace Chisholm (who, of course, then became Grace Chisholm Young) in 1896, and that Klein visited Cambridge to receive an honorary degree in 1897.

Together the Youngs, who formed a mathematical married partnership of real significance, left Cambridge and went to Göttingen. After a few months they went to Italy, together with the first of their six children (three sons and three daughters), where they lived for over a year. In September of 1899 they returned to Göttingen which was then home for them until 1908 when they moved to Geneva. However Young returned to Cambridge during term time where he both taught and examined. In 1913 he accepted two part-time chairs, one being the Hardinge Professorship of Pure Mathematics in Calcutta University which he held from 1913 to 1917, the other being at the University of Liverpool which he held from 1913 to 1919. He was the first to hold the Hardinge Professorship.

In 1915, while holding his two part-time chairs, the Youngs moved their permanent home from Geneva to Lausanne. This remained the family's permanent home even after 1919 when Young was appointed to the chair of mathematics at the University College of Wales in Aberystwyth in Wales. He held this post until 1923.

Young discovered a form of Lebesgue integration, independently but two years after Lebesgue. His definitions of measure and integration were quite different from those which Lebesgue had given but were shown to be essentially equivalent. He studied Fourier series and orthogonal series in general, the ideas which he put forward being further developed by LITTLEWOOD and HARDY.

Perhaps his most important contribution was to the calculus of several variables. He set out this theory beautifully in his treatise The fundamental theorems of the differential calculus (1910). All advanced calculus books now use his approach to functions of several complex variables. This 1910 book was one of three which Young wrote. The other two were written jointly with his wife: The first book of geometry (1905) was an elementary work clearly written by the Youngs with teaching mathematics to their own children in their minds, and The theory of sets of points (1906). Young was trapped in Lausanne when France fell in 1940. He was forced to spend the last two years of his life there very unhappy at being separated from his family. He received many honours for his mathematical achievements despite his lack of success in obtaining prestigious chairs. The was elected a Fellow of the Royal Society on 2 May 1907, receiving the Sylvester medal from that Society in 1928. He was president of the London Mathematical Society from 1922 to 1924 and president of the International Union of Mathematicians from 1929 to 1936. He received honorary degrees from the universities of Calcutta, Geneva, and Strasbourg.

†† **René Maurice Fréchet (1878-1973)** was born on 2 Sept 1878 in Maligny, Yonne, Bourgogne, France and died on 4 June 1973 in Paris, France. Maurice Fréchet was a student of HADAMARD's and, under his supervision, Fréchet wrote an outstanding dissertation in 1906, introducing the concept of a metric space. He did not invent the name 'metric space' which is due to HAUSDORFF. A versatile mathematician, Fréchet served as professor of mechanics at the University of Poitiers (1910-19) and professor of higher calculus at the University of Strasbourg (1920-27).

He held several different positions in the field of mathematics at the University of Paris (1928-48) including lecturer of the calculus of probabilities, professor of differential and integral calculus and professor of the calculus of probabilities. Fréchet made major contributions to the topology of point sets and defined and founded the theory of abstract spaces. Fréchet also made important contributions to statistics, probability and calculus. In his dissertation of 1906, mentioned above, he investigated functionals on a metric space and formulated the abstract notion of compactness. In 1907 he discovered an integral representation theorem for functionals on the space of quadratic Lebesgue integrable functions. A similar result was discovered independently by RIESZ.