MA-302 Advanced Calculus

10. Submanifolds

Felix Christian Klein†
(1849-1925)

The exercise 10.1 is only to get practice, its solution need not be submitted.

10.1. **a).** Let $F:\mathbb{R}^2\to\mathbb{R}$ be the function $F(x,y):=e^{x^2+2y^2+2}$. Find the level sets and determine which are manifolds.

b). For what values of the constant c is the locus of equation $\sin(x+y)=c$ is a smooth curve?

c). Show that if $c\neq 0$, then the hyperboloid $x^2+y^2-4z^2=c$ is a 2-manifold. Is the cone $x^2+y^2=4z^2$, a manifold? Find the tangent plane at $(2,-1,1)$ to the hyperboloid $x^2+y^2-4z^2=1$.

d). Show that $M:=\{(x,y,z)\in\mathbb{R}^3 \mid xy=0, x^2+y^2+z^2=1, z\neq\pm 1\}$ is a 1- manifold.

e). Show that $M:=\{(x,y)\in\mathbb{R}^2 \mid x^y=y^x\ ,\ x>0, y>0, (x,y)\neq(e,e)\}$ is a 1- manifold, where e is the Euler's number (is the base for the natural logarithms).

f). Is the subset $M:=\{(x,y,z)\in\mathbb{R}^3 \mid xy=xz=0\}$ a manifold?

g). Let $f,g:U\to\mathbb{R}$ be two C^k-functions on an open subset $U\subseteq\mathbb{R}$. Show that the subset $M:=\{(x,f(x),g(x))\in\mathbb{R}^3 \mid x\in U\}$ is a manifold and $(1,f'(x),g'(x))$ is a tangent vector to M at $(x,f(x),g(x))$.

h). (Orthogonal group) The set $\mathrm{O}(n,\mathbb{R}):=$ of

i). Show that the set $\mathrm{SO}(n)$ of all rotations of $\mathbb{R}^n$ about the origin 0 is an open in the set $\mathrm{O}(n)$ of all ortigonal transformations of $\mathbb{R}^n$ and hence $\mathrm{SO}(n)$ is a manifold of dimension $\binom{n}{2}$. Further, show that $\mathrm{SO}(n)$ is the largest connected subset of $\mathrm{O}(n)$ which contain the identity $\mathrm{id}_{\mathbb{R}^n}$.

10.2. Let $M\subseteq\mathbb{K}^m$ and let $f:\mathbb{K}^n\to\mathbb{K}^m$ be the affine linear map of the form $F(x)=\mathfrak{A}x+a$, where $\mathfrak{a}\in\mathrm{M}_{m,n}(\mathbb{K})$ of rank m and $a\in\mathbb{K}^m$.

a). If M is a smooth manifold of dimension d, then show that the inverse image $f^{-1}(M)$ is a smooth manifold of dimension $d+n-m$.

b). If $m=n$, then show that the direct image of M is also a smooth manifold of dimension d .

10.3. Let V,W be finite dimensional $\mathbb{K}$-vector spaces of dimensions m,n respectively. For $t\leq\min\{m,n\}$, let $F_t(V,W)$ denote the set of all $\mathbb{K}$-linear maps $f:V\to W$ of rank t . Then: $F_t(V,W)$ is a analytic $\mathbb{K}$-submanifold of $\mathrm{Hom}_{\mathbb{K}}(V,W)$ of dimension $t(M+n-t)$ and hence of codimension $(n-t)(m-t)$. The set $H_t(V,W)$ of all $\mathbb{K}$-linear maps of V into W of rank $\leq t$ is the disjoint union $H_t(V,W)=\biguplus_{t'\leq t}F'_t(V,W)$ of locally closed submanifolds $F'_t(V,W)$ $t'\leq t$. Further, the subset $F_t(V,W)$ is dense in $H_t(V,W)$.

10.4. (Stiefel-Manifolds) Let $m,n\in\mathbb{N}$, $m\leq n$. The real Stiefel-Manifold $\mathrm{St}_{\mathbb{R}}(m,n)$ is the set of all orthonormal m-tuples of (column) vectors from $\mathbb{R}^n$ (with the standard inner product). The elements of $\mathrm{St}_{\mathbb{R}}(m,n)$ are matrices from $\mathrm{M}_{n,m}(\mathbb{R})$. Therefore a matrix $\mathfrak{X}\in\mathrm{M}_{n,m}(\mathbb{R})$ belongs to $\mathrm{St}_{\mathbb{R}}(m,n)$ if and only if ${}^t\mathfrak{X}\mathfrak{X}=\mathfrak{E}_m$. This shows that $\mathrm{St}_{\mathbb{R}}(m,n)$ is the fibre $F^{-1}(\mathfrak{E}_m)$ of the map $F:\mathfrak{X}\mapsto{}^t\mathfrak{X}\mathfrak{X}$ from $\mathrm{M}_{n,m}(\mathbb{R})$ into the space of all real symmetric $m\times m$-matrices. Show that $\mathfrak{E}_m$ is a regular value of this map

10.5. (Matrices of fixed rank) For a fixed r, let $\mathrm{M}^r_{m,n}(\mathbb{K})$ denote the subset of $\mathrm{M}_{m,n}(\mathbb{K})$ of rank r. Show that if $0 \le r \le \min(m,n)$, then $\mathrm{M}^r_{m,n}(\mathbb{K})$ is an embedded submanifold [1]) of codimension $(m-r)(n-r)$ in $\mathrm{M}_{m,n}(\mathbb{K})$. (**Hint:** Let U be the open subset $\left\{\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix} \in \mathrm{M}_{m,n}(\mathbb{K}) \mid \det \mathfrak{a} \neq 0\right\}$ and let $F : U \to \mathrm{M}_{m-r,n-r}(\mathbb{K})$ be the map $\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{pmatrix} \mapsto \mathfrak{d} - \mathfrak{c}\mathfrak{a}^{-1}\mathfrak{b}$. Show that f is a submersion.)

10.6. Let $\Phi : V \times V \to \mathbb{K}$ be a non-degenerate symmetric $\mathbb{K}$-bilinear form on the $(n+1)$-dimensional $\mathbb{K}$-vector space V with the corresponding quadratic form $Q(x) = \Phi(x,x)$. In the case $\mathbb{K} = \mathbb{R}$ assume that Φ not negative definate. Show that the quadrik $M = \{x \in V \mid Q(x) = 1\}$ is a C^ω-hypersurface in V. The tangent space $\mathrm{T}_v M$ at a point $v \in M$ (as a subspace of V), is the orthogonal complement of $\mathbb{K}v$ with respect to Φ. In the case $\mathbb{K} = \mathbb{R}$, M is C^ω-diffeomorphic to $S^{p-1} \times \mathbb{R}^q$, if Φ is of type (p,q).

10.7. a). A product $S^{n_1} \times \cdots \times S^{n_r}$ of spheres of positive dimension is C^ω-diffeomorphic to a hypersurface in $\mathbb{R}^{n+1}$, $n := n_1 + \cdots + n_r$ More generally, the product of a 1-codimensional C^k-submanifold in $\mathbb{R}^{m+1}$ with a sphere S^n is diffeomorphic to a 1-codimensional C^k-submanifold of $\mathbb{R}^{m+n+1}$.

b). Let $a_1, \ldots, a_{n+1} \in \mathbb{N}^*$, $n \ge 1$. Then show that

$$M := \{x_1^{2a_1} + \cdots + x_{n+1}^{2a_{n+1}} = 1\} \subseteq \mathbb{R}^{n+1}$$

is a C^ω-hypersurface in $\mathbb{R}^{n+1}$ and $x \mapsto x/\|x\|$ is a C^ω-diffeomorphism of M onto S^n. What is the inverse of this diffeomorphism?

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† **Felix Christian Klein (1849-1925)** was born on 25 April 1849 in Düsseldorf, Prussia (now Germany) and died on 22 June 1925 in Göttingen, Germany. Felix Klein is best known for his work in non- euclidean geometry, for his work on the connections between geometry and group theory, and for results in function theory. Klein attended the Gymnasium in Düsseldorf. After graduating, he entered the University of Bonn and studied mathematics and physics there during 1865-1866. He started out on his career with the intention of becoming a physicist. While still studying at University of Bonn, he was appointed to the post of laboratory assistant to PLÜCKER in 1866. Plücker held a chair of mathematics and experimental physics at Bonn but, by the time Klein became his assistant, Plücker's interests had become very firmly rooted in geometry. Klein received his doctorate, which was supervised by Plücker, from the University of Bonn in 1868, with a dissertation Über die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien- Koordinaten auf eine kanonische Form. In his dissertation Klein classified second degree line complexes using Weierstrass's theory of elementary divisors.

However in the year Klein received his doctorate Plücker died leaving his major work on the foundations of line geometry incomplete. Klein was the obvious person to complete the second part of Plücker's Neue Geometrie des Raumes and this work led him to become acquainted with Clebsch. Clebsch moved to Göttingen in 1868 and, during 1869, Klein made visits to Berlin and Paris and Göttingen. In July 1870 Klein was in Paris when Bismarck, the Prussian chancellor, published a message which infuriated the French government. France declared war on Prussia on the 19th of July and Klein felt he could no longer remain in Paris and returned. Then, for a short period, he did military service as a medical orderly before being appointed as a lecturer at Göttingen in early 1871.

Klein was appointed professor at Erlangen, in Bavaria in southern Germany, in 1872. He was strongly supported by CLEBSCH, who regarded him as likely to become the leading mathematician of his day, and so Klein held a chair from the remarkably early age of 23. However Klein did not build a school at Erlangen where there were only a few students, so he was pleased to be offered a chair at the Technische Hochschule at Munich in 1875. There he, and his colleague BRILL, taught advanced courses to large numbers of excellent students and Klein's great talent at teaching was fully expressed. Among the students that Klein taught while at Munich were HURWITZ, VON DYCK, ROHN, RUNGE, PLANCK, BIANCHI and RICCI-CURBASTRO. Also in 1875 Klein married ANNE HEGEL, the granddaughter of the philosopher GEORG WILHELM FRIEDRICH HEGEL.

After five years at the Technische Hochschule at Munich, Klein was appointed to a chair of geometry at Leipzig. There he had as colleagues a number of talented young lecturers, including VON DYCK, ROHN, STUDY and ENGEL. The years 1880 to 1886 that Klein spent at Leipzig were in many ways to fundamentally change his life. Leipzig seemed to be a superb outpost for building the kind of school he now had in mind: one that would draw heavily on the abundant riches offered by Riemann's geometric approach to function theory. But unforeseen events and his always delicate health conspired against this plan. .. [In him were] two souls ... one longing for the tranquil scholar's life, the other for the active life of an editor, teacher, and scientific organiser. ... It was during the autumn of 1882 that the first of these two worlds came crashing down upon him ... his health collapsed completely, and throughout the years 1883-1884 he was plagued by depression.

His career as a research mathematician essentially over, Klein accepted a chair at the University of Göttingen in 1886. He taught at Göttingen until he retired in 1913 but he now sought to re-establish Göttingen as the foremost mathematics research centre in the

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[1]) This manifold is called Stiefel manifold and plays an important role in the study of principal fiber bundles.

world. His own role as the leader of a geometrical school at Leipzig was never transferred to Göttingen. At Göttingen he taught a wide variety of courses, mainly on the interface between mathematics and physics, such as mechanics and potential theory.

Klein established a research centre at Göttingen which was to serve as a model for the best mathematical research centres throughout the world. He introduced weekly discussion meetings, a mathematical reading room with a mathematical library. Klein brought HILBERT from Königsberg to join his research team at Göttingen in 1895.

The fame of the journal Mathematische Annalen is based on Klein's mathematical and management abilities. The journal was originally founded by Clebsch but only under Klein's management did it first rival, and then surpass in importance, Crelle's Journal. In a sense these journals represented the rival teams of the Berlin school of mathematics who ran Crelle's Journal and the followers of Clebsch who supported the Mathematische Annalen. Klein set up a small team of editors who met regularly and made democratic decisions. The journal specialised in complex analysis, algebraic geometry and invariant theory. It also provided an important outlet for real analysis and the new area of group theory.

Klein retired due to ill health in 1913. However he continued to teach mathematics at his home during the years of World War I.

It is a little hard to understand the significance of Klein's contributions to geometry. This is not because it is strange to us today, quite the reverse, it has become so much a part of our present mathematical thinking that it is hard for us to realise the novelty of his results and also the fact that they were not universally accepted by all his contemporaries.

Klein's first important mathematical discoveries were made in 1870 in collaboration with Lie. They discovered the fundamental properties of the asymptotic lines on the Kummer surface. Further collaboration with Lie followed and they worked on an investigation of W-curves, curves invariant under a group of projective transformations. In fact Lie played an important role in Klein's development, introducing him to the group concept which played a major role in his later work. It is fair to add that Camille Jordan also played a part in teaching Klein about groups. During his time at Göttingen in 1871 Klein made major discoveries regarding geometry. He published two papers On the So-called Non-Euclidean Geometry in which he showed that it was possible to consider euclidean geometry and non-euclidean geometry as special cases of a projective surface with a specific conic section adjoined. This had the remarkable corollary that non-euclidean geometry was consistent if and only if euclidean geometry was consistent. The fact that non-euclidean geometry was at the time still a controversial topic now vanished. Its status was put on an identical footing to euclidean geometry. Cayley never accepted Klein's ideas believing his arguments to be circular.

Klein's synthesis of geometry as the study of the properties of a space that are invariant under a given group of transformations, known as the Erlanger Programm (1872), profoundly influenced mathematical development. This was written for the occasion of Klein's inaugural address when he was appointed professor at Erlangen in 1872 although it was not actually the speech he gave on that occasion. The Erlanger Programm gave a unified approach to geometry which is now the standard accepted view.

Transformations play a major role in modern mathematics and Klein showed how the essential properties of a given geometry could be represented by the group of transformations that preserve those properties. In this way the Erlanger Programm defined geometry so that it included both Euclidean geometry and non-Euclidean geometry.

However Klein himself saw his work on function theory as his major contribution to mathematics. Klein considered his work in function theory to be the summit of his work in mathematics. He owed some of his greatest successes to his development of Riemann's ideas and to the intimate alliance he forged between the later and the conception of invariant theory, of number theory and algebra, of group theory, and of multidimensional geometry and the theory of differential equations, especially in his own fields, elliptic modular functions and automorphic functions.

By considering the action of the modular group on the complex plane, Klein showed that the fundamental region is moved around to tessellate the plane. In 1879 he looked at the action of PSL(2,7) thought of as an image of the modular group, and obtained an explicit representation of a Riemann surface. He showed it had equation $x^3y + y^3z + z^3x = 0$ given in homogeneous coordinates as a curve in the projective plane and its group of symmetries was PSL(2,7) of order 168. He wrote Riemanns Theorie der algebraischen Funktionen und ihre Integrale in 1882 which treats function theory in a geometric way connecting potential theory and conformal mappings. He also used physical ideas in this work, especially those of fluid dynamics.

Klein considered equations of degree greater than 4 and was particularly interested in using transcendental methods to solve the general equation of the fifth degree. After building on methods due to HERMITE and KRONECKER, producing similar results to Brioschi, he went on to completely solve the problem using the group of the icosahedron. This work led him to consider elliptic modular functions which he studied in a series of papers.

He developed a theory of automorphic functions, connecting algebraic and geometric results in his important 1884 book on the icosahedron. However POINCARÉ began publishing an outline of his theory of automorphic functions in 1881 and this led to a competition between the two. Klein initiated a correspondence with Poincaré, and soon a friendly rivalry ensued as both sought to formulate and prove a grand uniformization theorem that would serve as a capstone to this theory. Working under great stress, Klein succeeded in formulating such a theorem and in sketching a strategy for proving it.

However it was during this work that Klein's health collapsed as mentioned above. With Robert Fricke who came to Leipzig in 1884, Klein wrote a major four volume classic on automorphic and elliptic modular functions produced over the following 20 years.

We should also mention the Klein bottle, a one-sided closed surface named after Klein. A Klein bottle cannot be constructed in Euclidean space. It is best pictured as a cylinder looped back through itself to join with its other end. However this is not a continuous surface in 3-space as the surface cannot go through itself without a discontinuity. It is possible to construct a Klein bottle in non-Euclidean space.

In the 1890s Klein became interested in mathematical physics, although throughout his career he showed he was never far from this area in attitude. Following from this interest, he wrote an important work on the gyroscope with A SOMMERFELD. Later in his career Klein became interested in teaching at school level.

Starting in 1900 he began to take a lively interest in mathematical instruction below university level while continuing to pursue his academic functions. An advocate of modernizing mathematics instruction in Germany, in 1905 he played a decisive role in formulating the "Meraner Lehrplanentwürfe". The essential change recommended was the introduction in secondary schools of the rudiments of differential and integral calculus and the function concept.

Klein was elected chairman of the International Commission on Mathematical Instruction at the Rome International Mathematical Congress of 1908. Under his guidance the German branch of the Commission published many volumes on the teaching of mathematics at all levels in Germany.

Another project he worked on around the turn of the century was the Enzyklopädie der Mathematischen Wissenschaften. He took an active part in this project, editing with K Müller the four volume section on mechanics. Klein was elected a member of the Royal Society in 1885 and received the Copley medal of the Society in 1912.