Basic Algebra

3. Modules [1] — Submodules ;— Ideals

Max Noether†
(1844-1921)

Emmy Amalie Noether††
(1882-1935)

3.1. (Modular Law) For submodules U, W, N of an A–module V with $U \subseteq W$, prove that: $W \cap (U + N) = U + (W \cap N)$.

3.2. Let U, W, U', W' be submodules of an A–module V with $U \cap W = U' \cap W'$. Prove that U is the intersection of $U + (W \cap U')$ and $U + (W \cap W')$.

3.3. Let A be a ring and let V_i, $i \in I$, be an infinite family of non-zero A–modules. Prove that $W := \bigoplus_{i \in I} V_i$ is not a finite A–module.

3.4. For every natural number $m \geq 1$, give a minimal generating system for the $\mathbb{Z}$–module $\mathbb{Z}$ consisting of m elements.

3.5. Let K be a field and let A be a subring of K such that every element of K can be expressed as a quotient a/b with $a, b \in A$, $b \neq 0$. (i.e. K is the quotient field of A). If K is a finite A–module, then prove that $A = K$. In particular, $\mathbb{Q}$ is not a finite $\mathbb{Z}$–module. (**Hint:** Suppose $K = Ax_1 + \cdots + Ax_n$ and $b \in A$, $b \neq 0$, with $bx_i \in A$ for $i = 1, \ldots, n$. Now, try to express $1/b^2$ as a linear combination of x_i.)

3.6. Let K be a field and let V be a K–vector space. Suppose that $V_1, \ldots, V_n$ be distinct K-subspaces of V. If K has at least n elements (in particular, if K is infinite), then $V_1 \cup \cdots \cup V_n \neq V$. (**Hint:** Induction on n. By induction we may assume that $V_n \not\subseteq V_1 \cup \cdots \cup V_{n-1}$. Then there exist an elements $x \in V_n$, $x \notin V_1 \cup \cdots \cup V_{n-1}$ and $y \in V$, $y \notin V_n$. Now, consider the linear combinations $ax + y$, $a \in K$.)

3.7. (nil-radical) Let A be a *commutative* ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals in A. Show that $(\mathfrak{a} + \mathfrak{b})^m = \sum_{n=0}^{m} \mathfrak{a}^{m-n} \mathfrak{b}^n$ for all $m \in \mathbb{N}$. If $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are nilpotent ideals in A, then the sum ideal $\mathfrak{a}_1 + \cdots + \mathfrak{a}_r$ is also nilpotent. If $a_1, \ldots, a_r$ are nilpotent elements in A, then the ideal $Aa_1 + \cdots + Aa_r$ is nilpotent. The nilpotent elements form an ideal in A. (This ideal is called nil-radical of A and is denoted by $\mathfrak{n}_A$.)

3.8. Let $V \neq 0$ be an A–module, which does not have maximal submodules. Show that V does not have a minimal generating system.

3.9. Let A be a ring which is free from zero divisors. Suppose that the set of all non-zero left-ideals in A have a minimal element (with respect to the inclusion). Show that A is a division ring. In particular, a non-zero ring which is free from zero divisors in which the set of all left-ideals is artinian (with respect to inclusion), is a division ring.

[1]) The concept of a module seems to have made its first appearance in Algebra in *Algebraic Number Theory*– in studying subsets of *rings of algebraic integers*. Modules first became an important tool in Algebra in late 1920's largely due to the insight of EMMY NOETHER, who was the first to realize the potential of the module concept. In particular, she observed that this concept could be used to bridge the gap between two important developments in Algebra that had been going on side by side and independently:the theory of representations (=homomorphisms) of finite groups by matrices due to FROBENIUS, BURNSIDE, SCHUR et al and the structure theory of algebras due to MOLIEN, CARTAN, WEDDERBURN et al.

3.10. Let A be a non-zero ring and let I be an infinite indexed set. For every $i \in I$, let e_i be the I-tuple $(\delta_{ij})_{j\in I} \in A^I$ with $\delta_{ij} = 1$ for $j = i$ and $\delta_{ij} = 0$ for $j \neq i$.

a). e_i, $i \in I$, is a minimal generating system for the left-ideal $A^{(I)}$ in the ring A^I. In particular, $A^{(I)}$ is not finitely generated left-ideal. (**Remark:** Submodules of finitely generated modules need not be finitely generated!)

b). There exists a generating system for $A^{(I)}$ as an A^I–module that does not contain any minimal generating system. (**Hint:** First consider the case $I = \mathbb{N}$ and the tuples $e_0 + \cdots + e_n$, $n \in \mathbb{N}$.)

3.11. Let K_i, $i \in I$, be a family of fields. For every element $a = (a_i)_{i\in I} \in \prod_{i\in I} K_i$, let $\alpha(a)$ denote the zero-set $\{i \in I : a_i = 0\}$ of a. For an ideal $\mathfrak{a} \subseteq \prod_{i\in I} K_i$, let $\alpha(\mathfrak{a}) = \{\alpha(a) : a \in \mathfrak{a}\} \subseteq \mathfrak{P}(I)$. Show that: the map $\mathfrak{a} \mapsto \alpha(\mathfrak{a})$ is an isomorphism of the lattice [2]) of the ideals of $\prod_{i\in I} K_i$ onto the lattice of the filters [3]) defined on I. The ideal $\mathfrak{a}$ is maximal if and only if $\alpha(\mathfrak{a})$ is an ultra-filter [4]) on I. Deduce the following exercise [5]) from the Krull's theorem. Further, show that every finitely generated ideal in $\prod_{i\in I} K_i$ is a principal ideal.

Below one can see (simple) test-exercises.

Test-Exercises

T3.1. Let V be an A-module and let $a \in A$ be a unit. Then the homothecy $\vartheta_a : V \to V\ x \mapsto ax$ is bijective. Give an example of a non-zero A-module and a non-unit $a \in A$ such that the homothecy ϑ_a is bijective. (**Hint:** Consider $\mathbb{Z}$-modules.)

T3.2. Let A be a commutative ring and let e be an idempotent element in A. For every $a \in A$, show that $Aa \cap Ae = Aae$.

T3.3. Let A be a commutative ring and let $a, b \in A$ with $ab = 0$. Suppose that the ideal $Aa + Ab$ contain a non-zero divisor. Then show that $Aa \cap Ab = 0$, and that $a + b$ is a non-zero divisor in A.

T3.4. a). Let A_i, $1 \leq i \leq n$, be rings and let A be the product ring $\prod_{i=1}^n A_i$ and $p_i : A \to A_i$ be the canonical projections. Let $\mathfrak{a} \subseteq A$ be left–, right– resp. two sided ideal. Show that for every i, $\mathfrak{a}_i := p_i(\mathfrak{a})$ is a left–, right– resp. two sided ideal in A_i, and that $\mathfrak{a} = \prod_{i=1}^n \mathfrak{a}_i$. Conversely, if $\mathfrak{a}_i \subseteq A_i$, are left–, right – resp. two sided ideals, then show that $\mathfrak{a} := \prod_{i=1}^n \mathfrak{a}_i$ is a left–, right– resp. two sided ideal in A.

b). Let $A_1, \ldots, A_n$ be left– (resp. right–) principal ideal rings. Show that the direct product $\prod_{i=1}^n A_i$ ring is a left– (resp. right–) principal ideal ring.

c). All subrings of $\mathbb{Q}$ are principal ideal domains. (**Hint:** Let A be a subring of $\mathbb{Q}$. Show that $\mathfrak{a} = (\mathfrak{a} \cap \mathbb{Z})A$ for every ideal $\mathfrak{a}$ in A.)

T3.5. Let A be a ring, $\mathfrak{a}$ be a left ideal in A, which contain only nilpotent elements ($\mathfrak{a}$ need not be a nilpotent ideal!), and $\mathfrak{m}$ be an arbitrary maximal left- ideal in A. Show that $\mathfrak{a} \subseteq \mathfrak{m}$. (**Hint:** Consider $\mathfrak{a} + \mathfrak{m}$; if $1 = a + x \in \mathfrak{a} + \mathfrak{m}$ with $a \in \mathfrak{a}$ and $x \in \mathfrak{m}$, then $x = 1 - a \in A^\times$.)

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† **Max Noether (1844-1921)** Max Noether was born on 24 Sept 1844 in Mannheim, Baden, Germany and died on 13 Dec 1921 in Erlangen, Germany. Max Noether suffered an attack of polio when he was 14 years old and it left him with a

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[2]) **Lattice.** A partially ordered set $(X, \leq)$ is called a *lattice* if for every two elements $x, y \in X$, the supremum $\sup\{x, y\}$ and the infimum $\inf\{x, y\}$ exist. For example, the set of all left-ideals in a ring form a lattice with respect to the inclusion. What are $\sup\{\mathfrak{a}, \mathfrak{b}\}$ and $\inf\{\mathfrak{a}, \mathfrak{b}\}$ for left-ideals $\mathfrak{a}, \mathfrak{b}$ in A?

[3]) **Filter on a set.** Let X be any set and let $\mathfrak{P}(X)$ denote the power st of X. A *filter* on X is a subset $\mathfrak{F}$ of $\mathfrak{P}(X)$ such that: (1) $\mathfrak{F}$ is closed under finite intersections, i.e. intersection of finitely many elements of $\mathfrak{F}$ belongs to $\mathfrak{F}$. (In particular, the empty intersection $= X \in \mathfrak{F}$). (2) If $Y \in \mathfrak{F}$ and $Y \subseteq Z$, then $Z \in \mathfrak{F}$. Note that $\mathfrak{F} = \mathfrak{P}(X)$ if and only if $\emptyset \in \mathfrak{F}$.

[4]) **Ultra-filters on a set.** The set of filters on a set X is ordered by inclusion and it forms a lattice. Maximal elements in the set of filters on X different from $\mathfrak{P}(X)$ are called *ultra-filters* on X.

[5]) **Exercise.** Show that if $X \neq \emptyset$, then the set of of filters on X different from $\mathfrak{P}(X)$ is inductively ordered with respect to the inclusion and that every filter on X different from $\mathfrak{P}(X)$ is contained in an ultra-filter on X.

handicap for the rest of his life. He attended the University of Heidelberg from 1865 and obtained a doctorate from there in 1868. After this he lectured at Heidelberg and moved from Heidelberg to a chair at Erlangen where he remained for the rest of his life.

Max Noether was one of the leaders of nineteenth century algebraic geometry. He was influenced by Abel, Riemann, Cayley and Cremona. Following Cremona, Max Noether studied the invariant properties of an algebraic variety under the action of birational transformations. In 1873 he proved an important result on the intersection of two algebraic curves. Nine years later, in 1882, his daughter Emmy Noether was born. Emmy became interested in many similar topics to her father and generalised some of his theorems.

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† **Emmy Amalie Noether (1882-1935)** Emmy Amalie Noether was born on 23 March 1882 in Erlangen, Bavaria, Germany and died on 14 April 1935 in Bryn Mawr, Pennsylvania, USA. Emmy Noether's father Max Noether was a distinguished mathematician and a professor at Erlangen. Her mother was Ida Kaufmann, from a wealthy Cologne family. Both Emmy's parents were of Jewish origin and Emmy was the eldest of their four children, the three younger children being boys.

Emmy Noether attended the Höhere Töchter Schule in Erlangen from 1889 until 1897. She studied German, English, French, arithmetic and was given piano lessons. She loved dancing and looked forward to parties with children of her father's university colleagues. At this stage her aim was to become a language teacher and after further study of English and French she took the examinations of the State of Bavaria and, in 1900, became a certificated teacher of English and French in Bavarian girls schools. However Noether never became a language teacher. Instead she decided to take the difficult route for a woman of that time and study mathematics at university. Women were allowed to study at German universities unofficially and each professor had to give permission for his course. Noether obtained permission to sit in on courses at the University of Erlangen during 1900 to 1902. Then, having taken and passed the matriculation examination in Nürnberg in 1903, she went to the University of Göttingen. During 1903-04 she attended lectures by Blumenthal, Hilbert, Klein and Minkowski.

In 1904 Noether was permitted to matriculate at Erlangen and in 1907 was granted a doctorate after working under Paul Gordan. Hilbert's basis theorem of 1888 had given an existence result for finiteness of invariants in n variables. Gordan , however, took a constructive approach and looked at constructive methods to arrive at the same results. Noether's doctoral thesis followed this constructive approach of Gordan and listed systems of 331 covariant forms. Having completed her doctorate the normal progression to an academic post would have been the habilitation . However this route was not open to women so Noether remained at Erlangen, helping her father who, particularly because of his own disabilities, was grateful for his daughter's help. Noether also worked on her own research, in particular she was influenced by Fischer who had succeeded Gordan in 1911. This influence took Noether towards Hilbert's abstract approach to the subject and away from the constructive approach of Gordan.

Noether's reputation grew quickly as her publications appeared. In 1908 she was elected to the Circolo Matematico di Palermo, then in 1909 she was invited to become a member of the Deutsche Mathematiker Vereinigung and in the same year she was invited to address the annual meeting of the Society in Salzburg. In 1913 she lectured in Vienna.

In 1915 Hilbert and Klein invited Noether to return to Göttingen. They persuaded her to remain at Göttingen while they fought a battle to have her officially on the Faculty. In a long battle with the university authorities to allow Noether to obtain her habilitation there were many setbacks and it was not until 1919 that permission was granted. During this time Hilbert had allowed Noether to lecture by advertising her courses under his own name. For example a course given in the winter semester of 1916-17 appears in the catalogue as: **Mathematical Physics Seminar:** Professor Hilbert, with the assistance of Dr E Noether, Mondays from 4-6, no tuition.

Emmy Noether's first piece of work when she arrived in Göttingen in 1915 is a result in theoretical physics sometimes referred to as Noether's Theorem, which proves a relationship between symmetries in physics and conservation principles. This basic result in the general theory of relativity was praised by Einstein in a letter to Hilbert when he referred to Noether's penetrating mathematical thinking. It was her work in the theory of invariants which led to formulations for several concepts of Einstein's general theory of relativity. At Göttingen, after 1919, Noether moved away from invariant theory to work on ideal theory, producing an abstract theory which helped develop ring theory into a major mathematical topic. Idealtheorie in Ringbereichen (1921) was of fundamental importance in the development of modern algebra. In this paper she gave the decomposition of ideals into intersections of primary ideals in any commutative ring with ascending chain condition. Lasker (the world chess champion) had already proved this result for polynomial rings. In 1924 B L van der Waerden came to Göttingen and spent a year studying with Noether. After returning to Amsterdam van der Waerden wrote his book Moderne Algebra in two volumes. The major part of the second volume consists of Noether's work. From 1927 on Noether collaborated with Helmut Hasse and Richard Brauer in work on non- commutative algebras. In addition to teaching and research, Noether helped edit Mathematische Annalen. Much of her work appears in papers written by colleagues and students, rather than under her own name.

Further recognition of her outstanding mathematical contributions came with invitations to address the International Mathematical Congress at Bologna in 1928 and again at Zurich in 1932. In 1932 she also received, jointly with Artin, the Alfred Ackermann-Teubner Memorial Prize for the Advancement of Mathematical Knowledge. In 1933 her mathematical achievements counted for nothing when the Nazis caused her dismissal from the University of Göttingen because she was Jewish. She accepted a visiting professorship at Bryn Mawr College in the USA and also lectured at the Institute for Advanced Study, Princeton in the USA.

Weyl in his Memorial Address said: *Her significance for algebra cannot be read entirely from her own papers, she had great stimulating power and many of her suggestions took shape only in the works of her pupils and co-workers.*

van der Waerden writes: *For Emmy Noether, relationships among numbers, functions, and operations became transparent, amenable to generalisation, and productive only after they have been dissociated from any particular objects and have been reduced to general conceptual relationships.*