Basic Algebra

5. A. Structure Constants of an algebra – Quaternion algebras

Sir William Rowan Hamilton†
(1805-1865)

Using the following theorem in this exersise set we shall contruct algebras explicitly.

5.A.1 Theorem *Let A be a commutative ring, B a free A–module with A–basis x_i, $i \in I$, and e be an element in B. Suppose that the multiplication among basis elements of B is defined and extended to B by using the distributive law. This multiplication on B gives an A–algebra structure on B with the unit element e if and only if for all $i, j, k \in I$ we have* :

$$(x_i x_j)x_k = x_i(x_j x_k)\,, \quad ex_i = x_i = x_i e\,.$$

Moreover, if $x_i x_j = x_j x_i$ for all $i, j \in I$, then B is a commutative A–algebra.

PROOF The basis elements satisfy the desired associativity conditions. We have to show that the associative law for arbitrary elements $x = \sum_{i\in I} a_i x_i,\ y = \sum_{j\in I} b_j x_j,\ z = \sum_{k\in I} c_k x_k$ of B. Then using the distributive law, we have

$$(xy)z = \Big(\sum_i a_i x_i \sum_j b_j x_j\Big)\sum_k c_k x_k = \sum_{i,j} a_i b_j (x_i x_j) \sum_k c_k x_k = \sum_{i,j,k} a_i b_j c_k ((x_i x_j)x_k)$$
$$= \sum_{i,j,k} a_i b_j c_k (x_i(x_j x_k)) = \sum_i a_i x_i \sum_{j,k} b_j c_k (x_j x_k) = \sum_i a_i x_i \Big(\sum_j b_j x_j \sum_k c_k x_k\Big) = x(yz)\,.$$

The element e of B is the unit element of B, since from $ex_i = x_i = x_i e$ it follows that:

$$ex = e\Big(\sum_i a_i x_i\Big) = \sum_i a_i e x_i = \sum_i a_i x_i = x = \sum_i a_i x_i = \sum_i a_i x_i e = \Big(\sum_i a_i x_i\Big)e = xe\,.$$

In particular, the x_i is an algebra–generating system, the commutativity of B is ensured if the x_i are pairwise commutative. •

5.A.2 Remark The proof of 5.A.1 shows that the following more general assertion: Let A be a commutative ring and let B be an A-algebra in a general sense [1]) with an A–module–generating system

[1]) **Algebras in general sense** Let B be an A–algebra. Then B is a ring and hence the multiplication is associative. Further B has an unit element. Therefore more precisly we say that B is a associative A-algebra with unit element. (Earlier such algebras were also called hypercomplex systems over A (with unity.) Frequently the concept of algebra is used in more general sense and by an A-algebra in general sense we mean an A–module B together with a multiplication on B, for which the distributive laws $(x+y)z = xz+yz,\ z(x+y) = zx+zy,\ x, y, z \in B$, hold and for which the compatibility condition $(ax)(by) = ab(xy),\ a, b \in A,\ x, y \in B$ is fulfilled.

In other words an algebra over a commutative ring A is an A–module B with an A-bilinear map $B \times B \to B$. The A–bilinear map $B \times B \to B$ is called the multiplication of the A–algebra B and simply denoted by $(x, y) \mapsto xy$. If the multiplication in the A–algebra B is commutative (resp. associative, has an identity element (necessarily unique; this element is called the unit element of B)), then the A–algebra B is called commutative (resp. associative, unital or unitary).

Let B be an A–algebra; the maps $(x, y) \mapsto xy + yx$ and $(x, y) \mapsto xy - yx$ (with the A– module B) define two A–algebra structures on B, whihc are not in general associative; the first law is always commutative.

$x_i,\ i \in I$. Further, let $e \in B$. Suppose that $(x_i x_j)x_k = x_i(x_j x_k)$ for all $i, j, k \in I$ and $ex_i = x_i = x_i e$ for all $i \in I$, then B is associative with the unit element e.

5.A.3 Structure Constants Let A be a commutative ring and let B be a free A–algebra with the A–module basis $x_i,\ i \in I$. There exists an uniquely determined family

$$\gamma_{ij}^k\,,\quad (i, j, k) \in I \times I \times I\,,$$

of elements from A such that

$$x_i x_j = \sum_{k \in I} \gamma_{ij}^k\, x_k,\quad i, j \in I\,.$$

For fixed i, j, $\gamma_{ij}^k = 0$ for almost all $k \in I$. Note that k is an index and not a power. The coef|ficients γ_{ij}^k are called the *structure constants* of the A– algebra B with respect to the basis $x_i,\ i \in I$.

Conversely, suppose that B is a free A–module with A–basis $x_i,\ i \in I$. When can a family γ_{ij}^k of elements from A, where $\gamma_{ij}^k = 0$ for almost all k and fixed i, j, by $x_i x_j := \sum_{k \in I} \gamma_{ij}^k x_k$ and by extending using the distributive law, define a multiplication on B which give an A–algebra structure on B? By 5.A.1 we only need to ensure the associativity conditions $(x_i x_j)x_k = x_i(x_j x_k),\ i, j, k \in I$, and the existence of an element $e = \sum a_j x_j \in B$ such that $ex_i = x_i = x_i e,\ i \in I$. After a direct computation, this mean that the family γ_{ij}^k satisfy the following conditions:

$$\sum_r \gamma_{ij}^r \gamma_{rk}^s = \sum_r \gamma_{jk}^r \gamma_{ir}^s \quad \text{for all} \quad i, j, k, s \in I \ \text{ and } \sum_j a_j \gamma_{ji}^k = \sum_j a_j \gamma_{ij}^k = \delta_{ki} \quad \text{for all} \quad k, i \in I\,,$$

where δ_{ki} is the Kronecker–Symbol. This algebra is commutative if and only if we further have

$$\gamma_{ij}^k = \gamma_{ji}^k \quad \text{for all} \quad i, j, k \in I\,.$$

For the concrete construction of an algebra B with basis x_i and structure constants γ_{ij}^k, we give the multiplication of the basis elements in the form of the following table

	$\cdots$	x_j	$\cdots$
$\vdots$		$\vdots$	
x_i	$\cdots$	$\sum_k \gamma_{ij}^k x_k$	$\cdots$
$\vdots$		$\vdots$	

and this table is called the *structure-table* of B with respect to $x_i,\ i \in I$.

5.A.4 Remark In this remark we shall indicate a generalisation of 5.A.1 which is useful for the construction of rings if the ground ring is not commutative. Let A be an arbitrary ring, B be a free A–(left–) module with A–Basis $x_i,\ i \in I$, and let e be an element of B. Suppose that a multiplication among basis elements of B is defined and extended to B by using the distributive law. By using $x_i x_j = \sum_k \gamma_{ij}^k x_k$ define structuire constants γ_{ij}^k which are the elements of the *center* of A. Further, suppose that $(x_i x_j)x_k = x_i(x_j x_k)$ and $ex_i = x_i = x_i e$ for all $i, j, k \in I$. *Then there is a multiplication in* B *such that* B *is a ring with the unit element* e. The fact that e is the unit element is not that trivial as in 5.A.1. Let $e = \sum_j \varepsilon_j x_j$. For every i, we have $x_i = ex_i \sum_j \varepsilon_j(x_j x_i) = \sum_{j,k} \varepsilon_j \gamma_{ji}^k x_k$. Therefore $\sum_j \varepsilon_j \gamma_{ji}^k = \delta_{ik}$ (Kroneckersymbol). Now, for every $a \in A$, we have $e(ax_i) = \sum_j \varepsilon_j a(x_j x_i) = \sum_{j,k} \varepsilon_j a \gamma_{ji}^k x_k = \sum_k (\sum_j \varepsilon_j \gamma_{ji}^k) a x_k = ax_i$. Therefore bu uning distributivity, it follows that e is a left unit element. It is trivial that e is a right element. The proof of associativity of the multiplication is similar to that of the proof in, since the elements γ_{ij}^k commute with the elements of A. The details are left to the reader.

5.11. (*Monoid algebras*) Let M be a (multiplicatively written) monoid with the unit element 1. Further, let A be a commutative ring. For every $\sigma \in M$, let e_σ denote the canonical basis element

$(\delta_{\tau,\sigma})_{\tau\in M}$ of $A^{(M)}$, where $\delta_{\tau,\sigma}$ is the Kronecker symbol. For $\sigma, \tau \in M$, we define $e_\sigma e_\tau := e_{\sigma\tau}$. The structure table in this case is nothing but the binary-operation-table of M in which $\sigma \in M$ is replaced by e_σ. It is clear that the hypothesis of 5.A.1 for the basis e_σ, $\sigma \in M$, and the element $e = e_1$ are fulfilled. Therefore the structure constants are 0 or 1 and hence belong to the center of A. Therefore, by 5.A.1 or 5.A.4 there is a ring structure on $A^{(M)}$ with the unit element e_1. The algebra so defined is called the Monoid algebra of M over A and is dnoted by $A[M]$. If M is a group, then this algebra is called the Group algebra of M ovr A. Special cases of the monoid algebras are polynomial rings. These are the algebras of the monoid $\mathbb{N}^{(I)}$, where I is an indexed set and the binary operation on $\mathbb{N}$ is the usual addition.

5.12. (Generalised quaternions over a commutative ring) Let A be a commutative ring and let $a, b \in A^\times$, we shall construct the A–algebra $(a,b)_A$ the (generalised) quaternions of type (a,b). We consider the free A–module A^4, the canonical basis of A^4 is denoted by $1, i, j, k$ and a multiplication on A^4 is defined by the structure-table

	1	i	j	k
1	1	i	j	k
i	i	a	k	aj
j	j	$-$k	b	$-b$i
k	k	$-a$j	bi	$-ab$

and extend by using distributive law. We leave the verification of the associativity conditions to the reader. Clearly 1 is the unit element of this multiplication. We shall identify A with $A \cdot 1$. The elements $z = c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k}$ of $(a,b)_A$ are called quaternions. The quaternions with $c_0 = 0$ are called pure. For a pure quaternions z, w we have: $zw = -wz$. Therefore $(a,b)_A$ is commutative if and only if either $A = 0$ or Char $A = 2$.

a). ((quaternion–)conjugation) For each quaternion $z = c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k} \in (a,b)_A$, the quaternion $\overline{z} := c_0 - c_1\mathrm{i} - c_2\mathrm{j} - c_3\mathrm{k}$ is called the conjugate quaternion of z. For a pure quaternion z we have : $\overline{z} = -z$. Further, for $z, w \in (a,b)_A$, we have $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{w}\,\overline{z}$ and $\overline{\overline{z}} = \overline{z}$. If $2 \in A^*$, then $z = \overline{z}$ if and only if $z \in A$. In particular, the map $z \mapsto \overline{z}$ is an anti-automorphism of the A-algebra $(a,b)_A$ and is called the (quaternion–)conjugation. (**Proof.** Clearly $\overline{cz} = c\overline{z} = \overline{z}c$ for every $c \in A$. In view of the given compatibility with addition, it is enough to prove the assertion for $z, w \in \{\mathrm{i},\mathrm{j},\mathrm{k}\}$. For example $\overline{\mathrm{i}\cdot\mathrm{i}} = \overline{a} = a = (-\mathrm{i})\cdot(-\mathrm{i}) = \overline{\mathrm{i}}\cdot\overline{\mathrm{i}}$ und $\overline{\mathrm{i}\cdot\mathrm{j}} = \overline{\mathrm{k}} = -\mathrm{k} = \mathrm{j}\cdot\mathrm{i} = (-\mathrm{j})\cdot(-\mathrm{i}) = \overline{\mathrm{j}}\cdot\overline{\mathrm{i}}$. The remaining part is left to the reader.)

b). ((Reduced) Norm) For each quaternion $z \in (a,b)_A$, the element $\mathrm{N}(z) := z\overline{z}$ is called the (reduced) norm of z. For $z = c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k}$, $c_0, c_1, c_2, c_3 \in A$, we have

$$\mathrm{N}(z) = c_0^2 - ac_1^2 - bc_2^2 + abc_3^2\,,$$

For $z, w \in (a,b)_A$, we have $\mathrm{N}(z) = \mathrm{N}(\overline{z})$ and $\mathrm{N}(zw) = \mathrm{N}(z)\mathrm{N}(w)$. In particular, the norm is a multiplicative map $\mathrm{N} : (a,b)_A \to A$. (**Proof.** We have $\mathrm{N}(zw) = zw\,\overline{zw} = zw\overline{w}\,\overline{z} = z\mathrm{N}(w)\overline{z} = \mathrm{N}(z)\mathrm{N}(w)$. This where we have used the fact that $\mathrm{N}(w)$ is contained in the center of $(a,b)_A$.)

c). A quaternion $z \in (a,b)_A$ is a unit in $(a,b)_A$ if and only if $\mathrm{N}(z)$ is a unit in A. In this case $z^{-1} = \mathrm{N}(z)^{-1}\overline{z}$. (**Proof.** Since $zw = wz = 1$, we have $1 = \mathrm{N}(1) = \mathrm{N}(zw) = \mathrm{N}(z)\mathrm{N}(w)$. Conversely, suppose that $\mathrm{N}(z)$ is a unit in A. Then $\mathrm{N}(z)^{-1}$ commutes with all quaternions and $\mathrm{N}(z)^{-1}\overline{z}$ is the inverse of z, since $1 = z(\mathrm{N}(z)^{-1}\overline{z}) = (\mathrm{N}(z)^{-1}\overline{z})z$.)

d). The A–algebra $(a,b)_A$ is a division ring if and only if A is a field and $\mathrm{N}(z) = 0$, $z \in (a,b)_A$, only for $z = 0$. The last condition means that the only solution of the equation $c_0^2 - ac_1^2 - bc_2^2 + abc_3^2 = 0$ in A is the trivial solution $c_0 = c_1 = c_2 = c_3 = 0$.

e). The A-algebra $(1,1)_A$ is not a division ring, since the reduced norm is $\mathrm{N}(z) = c_0^2 - c_1^2 - c_2^2 + c_3^2$ and $\mathrm{N}(z) = 0$ has the non-trivial solution $(1,1,1,1)$. Similarly, the A-algebras $(1,-1)_A$ and $(-1,1)_A$ are not division rings.

f). For an algebraically closed field K the generalised quaternion algebras are not division rings. In particular, $\mathbb{H}(\mathbb{C}) := (-1,-1)_{\mathbb{C}}$ is not a division ring. (**Hint:** For $c = \mathrm{i} \in \mathbb{C}$ and $0 \neq z := 1 + c\mathrm{i} \in \mathbb{H}(\mathbb{C})$, the norm $\mathrm{N}(z) = 0$. More precisely, we have the following interesting theorem (which need some more preparation!)

5.A.5 Theorem *Let k be a field of* $\operatorname{Char} k \neq 2$ *and let $a, b \in k^\times$. Then*

a). *$(a,b)_k$ is a simple k–algebra with center k.*

b). *If $(a,b)_k$ is not a division ring, then $(a,b)_k$ is isomorphic to the k–algebra $\mathrm{M}_2(k)$ of (2×2)–matrices over k.*

c). *If k is algebraically closed, then $(a,b)_k$ is isomorphic to the k–algebra $\mathrm{M}_2(k)$ of (2×2)–matrices over k.*

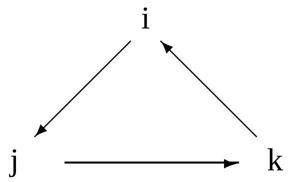
g). Let K be a finite field. Then the quaternion algebra $(a,b)_K$, $a, b \in K^\times$ is not a divison ring. (**Hint:** It is enough to prove that: for given $a, b, c \in K^\times$, there exist $x, y \in K$ such that $ax^2 + by^2 = c$. Let $q := |K|$, $M := \{ax^2 : x \in K\}$ and $N := \{c - by^2 : y \in K\}$. Then $|M| = q$, if q is odd (use the exercise[2])) and $|M| = (q+1)/2$, if q even (use the exercise[3])). In any case $|M| = |N|$ and so $M \cap N \neq \emptyset$.)

h). ((Reduced) Trace) For a quaternion $z \in \mathbb{H}(A)$, A commutative ring, the element $z + \overline{z} \in A$ is called the (reduced) trace of z and is denoted by $\mathrm{Tr}(z)$. For $a, b \in A$, $z, w \in (a,b)_A$, we have $\mathrm{Tr}(z) = \mathrm{Tr}(\overline{z})$ and $\mathrm{Tr}(az + bw) = a\mathrm{Tr}(z) + b\mathrm{Tr}(w)$. In particular, the trace is a A–linear $\mathrm{Tr} : (a,b)_A \to A$. Further, $z^2 - \mathrm{Tr}(z)z + \mathrm{N}(z) = 0$ for all $z \in \mathbb{H}(A)$.

5.13. For a commutative ring A, in the special case $a = b = -1$, we denote the A-algebra $(-1,-1)_A$ by $\mathbb{H}(A)$. [4]) This is the classical quaternion algebra over A; its structure-table is:

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1 .

The multiplication of the elements i, j, k is listed according to the following scheme: First write these elements in the form



Then, if x, y, z are arbitrary succesive three elements in $\{\mathrm{i},\mathrm{j},\mathrm{k}\}$, then $xy = z$, in the case the diagramm contains $x \to y$ and $xy = -z$ otherwise. Further, if $x^2 = -1$ for $x \in \{\mathrm{i},\mathrm{j},\mathrm{k}\}$. For $z = c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k}$, we have $\mathrm{N}(z) = c_0^2 + c_1^2 + c_2^2 + c_3^2$.

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[2]) **Exercise** Let G be a finite group of order m and let $n \in \mathbb{Z}$. Then $\gcd(m,n) = 1$ if and only if the map $G \to G$ defined by $x \mapsto x^n$ is bijective.

[3]) **Exercise** Let G be a finite group of even order $m = 2n$. Then there are exactly n elements of G which are squares in G. (**Hint:** Look at the bibers of the map $G \to G$, $x \mapsto x^2$.

[4]) The letter "$\mathbb{H}$" is used to denote this quaternion algebra as this algebra was first discovered by HAMILTON in 1843. This was one of the first non-commutative ring discovered. Further, this is is a division ring, was extremely influential in the subsequent developemnet of mathematics and it continues to play an important role in certain areas of mathematics and physics; However, it believed that the quaternions were known to EULER, GAUSS and others before. Later HURWITZ, A. (1859-1919) had considered quaternion algebra over the ring $\mathbb{Z}$ of integers. This algebra was used to prove the famous classical theorem of LAGRANGE, J. L. (1736-1813) on the sum of four square theorem: *Every integer can be expressed as the sum of squares of four integers*. This theorem was the starting point of a large research area in number theory, so called Waring problem. This asks *if every integer can be expressed as a sum of fixed number of k-th powers*. For instance it can be ashown that every integer is a sum of nine cubes (resp. nineteen 4-th powers, etc...). HILBERT had shown that the Waring problem have an affirmative answer.

a). If A is a subfield of $\mathbb{R}$, then clearly $\mathrm{N}(z) = 0$ is equivalent to $c_0 = c_1 = c_2 = c_3 = 0$ and $\mathbb{H}(A)$ is a division ring. In particular, $\mathbb{H}(\mathbb{Q})$ and the " usual " quaternions $\mathbb{H} := \mathbb{H}(\mathbb{R})$ are division rings.

b). In the quaternion algebra $\mathbb{H}(\mathbb{Z})$ over $\mathbb{Z}$ the units $z = c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k}$ are determined by the condition $\mathrm{N}(z) = c_0^2 + c_1^2 + c_2^2 + c_3^2 \in \mathbb{Z}^\times = \{1, -1\}$. Therefore the unit group $\mathbb{H}(\mathbb{Z})^\times$ the group of eight elements ± 1, $\pm\mathrm{i}$, $\pm\mathrm{j}$, $\pm\mathrm{k}$. This very interesting unit group is called the quaternion group.

c). In the quaternion algebra $\mathbb{H}(A)$, the quaternions of norm 1 forms a subgroup of the unit group $\mathbb{H}(A)^\times$ of $\mathbb{H}A$. In the case $A = \mathbb{R}$ this group is called the Spin–Group.

5.14. In this we list many interesting results with out proofs. First the two interesting theorems 5.A.6 and 5.A.7 (which need some more preparation!) are due to WEDDERBURN, H. M. (1882-1948) and FROBENIUS, G. (1849-1917) respectively.

5.A.6 Theorem (Wedderburn, 1905) *Every finite division ring is commutative and hence a field.*

5.A.7 Theorem (Frobenius, 1877) $\mathbb{R}$, $\mathbb{C}$ *and* $\mathbb{H}$ *are the only non-isomorphic division* $\mathbb{R}$*–algebras which are finite dimensional over* $\mathbb{R}$.

5.A.8 Octonions Let a, b be non-zero elements of a field K and let $H := (a, b)_K$ be the quaternion algebra of type (a, b) over K (see exercise 5.12) with the standard basis $1, \mathrm{i}, \mathrm{j}, \mathrm{k} = \mathrm{i}\,\mathrm{j}$.

For an arbitrary element $c \in K$, on $O := H \times H$ define a K-bilinear multiplication by

$$(\omega_1\,, \eta_1)\,(\omega_2\,, \eta_2) := (\omega_1\omega_2 + c\,\overline{\eta}_2\eta_1\,, \eta_2\omega_1 + \eta_1\overline{\omega}_2)$$

and the conjugation by

$$\overline{(\omega\,, \eta)} := (\overline{\omega}\,, -\eta)\,.$$

We identify H with $H \times \{0\}$ in O, and hence the multiplication on O can be restricted to H, this multiplication is in genearl *not* associative. The element $1 \in H \subseteq O$ is also the unit element for O. Further, $\overline{(\omega_1\,, \eta_1) \cdot (\omega_2\,, \eta_2)} = \overline{(\omega_2\,, \eta_2)}\;\overline{(\omega_1\,, \eta_1)}$. The quadratic form

$$\mathrm{N}(\omega\,, \eta) := (\omega\,, \eta)\,\overline{(\omega\,, \eta)} = \omega\,\overline{\omega} - c\,\overline{\eta}\eta = \mathrm{N}(\omega) - c\,\mathrm{N}(\eta)$$

on O is called the Norm. This norm is multiplicative:

$$\mathrm{N}\big((\omega_1\,, \eta_1)\,(\omega_2\,, \eta_2)\big) = \mathrm{N}(\omega_1\,, \eta_1) \cdot \mathrm{N}(\omega_2\,, \eta_2)\,.$$

Therefore it follows that : *If the norm* $\mathrm{N}(\omega, \eta) = \mathrm{N}(\omega) - c\,\mathrm{N}(\eta)$ *on* O *is anisotrop, then the corresponding multiplication of* O *is free from zero divisors.* This K–algebra $O = \mathsf{O}_K(a, b, c)$ is (which is in general not associative) the well-known algebra of Cayley-numbers or the octonion algebra of type (a, b, c) over K; this is 8-dimensional algebra over K. The algebra $\mathbb{O}(K) := \mathsf{O}_K(-1, -1, -1)$ is called the octonion algebra. In the case $K = \mathbb{R}$, this algebra is simply denoted by $\mathbb{O}$. Its norm is positive definite. *Therefore* $\mathbb{O}$ *is free from zero divisors* and hence a division algebra in general sense (see footnote [1)]), in which the equations $xy = z$ for given y, z with $y \neq 0$ resp. for given x, z with $x \neq 0$ has exactly one solution x resp. y.

5.A.9 Theorem (Adam, Bott-Milnor, Kervaire) *If* $n \neq 0, 1, 2, 4, 8$, *then there does not exist an* n*–dimensional real division algebra in general sense.*

This is a very deep theorem and was first proved by ADAMS by using *Topological* K*-theory*. Nowadays there is a proof by BOTT, MILNOR and KERVAIRE. (see Milnor, J.: Some Consequences of a Theorem of Bott. Ann. Math. **68** (1958) 444–449.)

5.A.10 Real Clifford algebras[5)] The sequence of real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ can be extended by introducing the (real) Clifford algebras Cl_n for $n \geq 0$.

[5)] The study of real Clifford algebras have become central in *modern geometry and topology*; they also appear in *Quantum Theory* in connection with the *Dirac operator*. The groups of units in Clifford algebras contain the *spinor groups*; they provide double coverings of the *special orthogonal groups*. We have only given definition of real Clifford algebras associated to positive definite inner products on $\mathbb{R}^n$. There are also complex and indefinite Clifford algebras.

For each integer $n \geq 0$, the Clifford algebra[6]) Cl_n is the associative algebra over $\mathbb{R}$ that is generated (as an $\mathbb{R}$–algebrra) by a unity 1 and elements $e_1, e_2, \ldots, e_n$, subject only to relations $e_i^2 = -1$, $e_i e_j = -e_j e_i$ for $i \neq j, 1 \leq i, j \leq n$. It is evident that $\mathrm{Cl}_0 = \mathbb{R}$ viewed as 1-dimensional algebra over it self. The algebra Cl_1 is generated by 1 and e_1 subject only to relation $e_1^2 = -1$. But this is just the way that the complex numbers are described when viewed as a 2-dimensional algebra over $\mathbb{R}$. Therefore if we set $e_1 = \mathrm{i}$, then $\mathrm{Cl}_1 = \mathbb{C}$. The quternion algebra $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$ generated by 1, i, j, k with relations $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$, $\mathrm{jk} = -\mathrm{kj} = \mathrm{i}$ and $\mathrm{ki} = -\mathrm{ik} = \mathrm{j}$. Therefore if we set $e_1 = \mathrm{i}$, $e_2 = \mathrm{j}$ and $e_1 e_2 = \mathrm{k}$, it is easy to cheak that $\mathrm{Cl}_2 = \mathbb{H}$. All of the Clifford algebras have been explicitly computed [7]) by ATIYAH, BOTT and SHAPIRO. For $0 \leq n \leq 7$ the Clifford algebras are given in the following table and all other Clifford algebras can now be computed by using the isomorphism $\mathrm{Cl}_{n+8} \cong \mathrm{M}_{16}(\mathrm{Cl}_n)$. The mysterious numbers a_{n+1} in the 4-th row of the table are similarly defined for all indices by setting $a_{(n+1)+8} = 16 a_{n+1}$. For each $n \geq 0$, the Clifford algebra Cl_n has a representation on $\mathbb{R}^{a_n}$. All these results can be proved by using *tensor products of algebras*, the prrofs are given in [loc.cit.] and are quite accessible and elegant. The first few real Clifford algebras are summarised in the following table :

n	0	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$\mathrm{M}_2(\mathbb{H})$	$\mathrm{M}_4(\mathbb{C})$	$\mathrm{M}_8(\mathbb{R})$	$\mathrm{M}_8(\mathbb{R}) \times \mathrm{M}_8(\mathbb{R})$	$\mathrm{M}_{16}(\mathbb{R})$
$\mathrm{Dim}_{\mathbb{R}}\mathrm{Cl}_n$	1	2	4	8	16	32	64	128	256
a_{n+1}	1	2	4	4	8	8	8	8	8

5.15. On $\mathbb{C}^n$, $n \in \mathbb{N}$, $n \geq 2$, there does not exist a division $\mathbb{C}$–algebra structure in general sense.

(**Proof** This follows very easily from the *fundamental theorem of algebra*[8]) and hence its proof works for any algebraically closed [9]) field K. For the proof we shall use the determinant theory. In view of the exercise 5.10c), it is enough to show that there exists $z \in K^n$ such that the left multiplication map $\lambda_z : K^n \to K^n$ which is a K-linear endomorphism of K^n, is not bijective, i.e. the determinant $\mathrm{Det}\,\lambda_z$ of λ_z is equal to 0. For this let u and v be two linearly independent elements in K^n (e.g. $u := e_1$, $v := e_2$). We may assume that λ_u is bijectiv; otherwise take $z := u$. Then we consider for $t \in K$ the determinant of $\lambda_u^{-1} \circ \lambda_{u+tv} = \lambda_u^{-1}(\lambda_u + t\lambda_v) = \mathrm{id} + t\,\lambda_u^{-1}\lambda_v$. This is a polynomial function of degree n in t (namely, the value of the characteristic polynomial of $\lambda_u^{-1}\lambda_v$ at $-t$. Now, since K is algebraically closed, there exists $t_0 \in K$ such that $\mathrm{Det}(\mathrm{id} + t_0\lambda_u^{-1}\lambda_v) = 0$. But then λ_z for $z := u + t_0 v$ is not bijective.)

5.16. (Hurwitz's quaternions) The quaternion $\mathbb{Z}$–algebra $\mathbb{H}(\mathbb{Z})$ is a $\mathbb{Z}$–subalgebra of $\mathbb{H}(\mathbb{Q})$. An element $z = a + b_{\mathrm{i}} + c\mathrm{j} + d\mathrm{k} \in \mathbb{H}(\mathbb{Q})$ is called a Hurwitz's quaternion if $a = a'/2, b = b'/2, c = c'/2, d = d'/2$, where the numbers a', b', c', d' are either all even integers (in this case $z \in \mathbb{H}(\mathbb{Z})$) or all odd intgers. The Hurwitz's Quaternions form a non-commutative $\mathbb{Z}$–algebra H' between $\mathbb{H}(\mathbb{Z})$ and $\mathbb{H}(\mathbb{Q})$, it is a free $\mathbb{Z}$–algebra of rank 4. Find a $\mathbb{Z}$–basis of H'. If $z \in H'$, then $N(z) \in \mathbb{Z}$. Further, $z \in H'$ is a unit in H' if and only if $N(z) = 1$. The unit group

[6]) Clifford algebra were introduced WILLIAM KINGDON CLIFFORD (1845-1879). Clifford generalised the quaternions (introduced by Hamilton two years before Clifford's birth) to what he called the biquaternions and he used them to study motion in non-euclidean spaces and on certain surfaces. These are now known as 'Clifford-Klein spaces'. He showed that spaces of constant curvature could have several different topological structures.

[7]) See [Atiyah, M.F., Bott, R. and Shapiro, A., *Clifford modules*, Topology **3** (1964), 3-38.]

[8]) Fundamental theorem of algebra *Every non-constant polynomial with coefficients in* $\mathbb{C}$ *has a zero in* $\mathbb{C}$. The fundamental theorem of algebra was stated first by a French mathematician JEAN-DE-ROND D'ALEMBERT 91717-1855), who gave imcomplete proof. The first correct proof of this theorem was given by GAUSS in 1799.

[9]) A field K is called algebraically closed if the fundamental theorem of algebra holds for K, i.e. every polynomial of positive degree over K has a zero in K. For example, $\mathbb{C}$ is algebraically closed, but $\mathbb{Q}$ and $\mathbb{R}$ are not algebraically closed.

$H'^{\times}$ contains 24 elements, namely

$$\varepsilon,\ \varepsilon\mathrm{i},\ \varepsilon\mathrm{j},\ \varepsilon\mathrm{k},\ \frac{1}{2}(\varepsilon_0+\varepsilon_1\mathrm{i}+\varepsilon_2\mathrm{j}+\varepsilon_3\mathrm{k},\quad \text{mit}\ \varepsilon,\ \varepsilon_0,\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\in\{1,-1\}\,.$$

For every Hurwitz's quaternion $z\in H'$ there exists an unit $e\in H'^{\times}$ such that $ez\in\mathbb{H}(\mathbb{Z})$. (**Hint:** If $z\notin\mathbb{H}(\mathbb{Z})$, then there exists a $f\in H'^{\times}$ such that $z+f=2z'$ and $z'\in\mathbb{H}(\mathbb{Z})$. Now multiply by f^{-1} on the left.)

5.17. Let K be a field and let B be a non-zero finite K–algebra which is free from zero divisors. Then B and every K–subalgebra of B is a division ring. (**Proof** Since all K–subalgebras of B are finite and free from zero divisors, it is enough to prove that B is a division ring. Let $x_1,\dots,x_m$ be a K–basis of B. For an arbitrary element $x\in B,\ x\neq 0$, $xx_1,\dots,xx_m$ resp. $x_1x,\dots,x_mx$ are bases of B. For this it is enough to prove that these elements are linearly independent over K. Suppose that $a_1xx_1+\cdots+a_mxx_m=0$ with $a_1,\dots,a_m\in K$. Then $x(a_1x_1+\cdots+a_mx_m)=0$ and hence $a_1x_1+\cdots+a_mx_m=0$, since $x\neq 0$ and B is free from zero divisors. Therefore $a_1=\cdots=a_n=0$, similarly, $x_1x,\dots,x_mx$ are linearly independent over K. Now, let $z\in B$ be arbitrary. Then there exist elements $b_1,\dots,b_m$ resp. $c_1,\dots,c_m$ in K such that $z=b_1xx_1+\cdots+b_mxx_m=x(b_1x_1+\cdots+b_mx_m)$ resp. $z=c_1x_1x+\cdots+c_mx_mx=(c_1x_1+\cdots+c_mx_m)x$. This shows that the equations $xu=z$ resp. $vx=z$ have solutions in B. Now use the exercise 5.10c).)

5.18. Let K be a field of Char $K\neq 2$ and let $a,b\in K^{\times}$. Then the following statements are equivalent:

(i) There exist $c_0,c_1,c_2,c_3\in K$, not all zero such that $c_0^2-a\,c_1^2-b\,c_2^2+abc_3^2=0$.

(ii) There exist $c_0,c_1,c_2\in K$, not all zero such that $c_0^2-ac_1^2-bc_2^2=0$.

(iii) There exist $c_1,c_2,c_3\in K$, not all zero such that $-ac_1^2-bc_2^2+abc_3^2=0$.

Below one can see (simple) test-exercises.

Test-Exercises

T5.12. Let K be a field. Construct a non-commutative K–algebra of dimension 3 with basis $1,z,w$ and the structure-table

	1	z	w
1	1	z	w
z	z	z	w
w	w	0	0 .

(**Remark:** This is the only non-commutative K–algebra of dimension 3.)

T5.13. The center of $\mathbb{H}(\mathbb{R})$ is $\mathbb{R}$. — What is the center of $\mathbb{H}(A)$ for an arbitray commutative ring A?

T5.14. **a).** Find all $z\in\mathbb{H}=\mathbb{H}(\mathbb{R})$ with $z^2=-1$. **b).** Let B be a $\mathbb{R}$–subalgebra of $\mathbb{H}$ with $\mathbb{R}\subset B\subset\mathbb{H}$. Show that : there exists a $z\in\mathbb{H}$ with $z^2=-1$ and $B=\mathbb{R}[z]$. What is the dimension B over $\mathbb{R}$? Compare the $\mathbb{R}$–algebra B with the $\mathbb{R}$–algebra $\mathbb{C}$ of the complex numbers. Let $z,w\in\mathbb{H}$ with $z^2=w^2=-1$. Show that $\mathbb{R}[z]=\mathbb{R}[w]$ if and only if $w=z$ or $w=-z$.

T5.15. Let A commutative ring and let $z\in\mathbb{H}(A)$ be a quaternion. Show that the follwoing statements are equivalent:

(i) z is a left zero-divisor in $\mathbb{H}(A)$. (ii) z is a right zero-divisor in $\mathbb{H}(A)$.

(iii) z is a zero-divisor in $\mathbb{H}(A)$. (iv) $\mathrm{N}(z)$ is a zero-divsior in A.

T5.16. Let $z\in\mathbb{H}(\mathbb{R})$. the non-negative square root $\sqrt{\mathrm{N}(z)}$ of the norm of z is called the *absolute value* of the quaternion z. For $z,w\in\mathbb{H}$, we have $|zw|=|z||w|$ and the *triangle inequality* $|z+w|\leq|z|+|w|$.

T5.17. Generalise 5.18 as follows : Let B be a ring which is free from zero-divisors. Suppose that B is finite as a left-vector space *or* as a right-vector space finite over a sub-divison ring K. Then B and every subring between K and B is a division ring.

T5.18. Generalise 5.18 as follows : Let K be a field and let $B \neq 0$ be a K–algebra in the general sense (see footnote [1)]). If $\mathrm{Dim}_K B$ is finite and if B is free from zero-divisors (i.e. $yz \neq 0$ if $0 \neq z$), then B is a division algebra in the general sense (see footenote [1)]), i.e. for given $u, v \in B,\ u \neq 0$, the equation $uw = v$ resp. $xu = v$ has exactly one solution w resp. x.

T5.19. Let K be a field and let M be a monoid generated by one element σ. Then $K[M] = K[e_\sigma]$ is a monogene K–algebra and $K[M]$ is a principal ideal ring. (see exercise 5.8)

T5.20. Let G be a group and let $g \in G$ be an element of finite order $m > 1$. Then the element $1-g = e_1 - e_g$ in the group ring $A[G]$ is a zero divisor. (**Hint :** $(1-g)(1+g+\cdots+g^{m-1}) = 1-g^m = 1-1 = 0$ in the ring $A[G]$.)

T5.21. Let $\mathrm{Q}_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion (multiplicatively written) group. Then the group ring $\mathbb{R}[\mathrm{Q}_8]$ of Q_8 over $\mathbb{R}$ is not same as $\mathbb{H}$ even though later conatin a copy of the quaternion group Q_8.

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† **Sir William Rowan Hamilton (1805-1865)** was born on 4 Aug 1805 in Dublin, Ireland and died on 2 Sept 1865 in Dublin, Ireland. William Rowan Hamilton's father, Archibald Hamilton, did not have time to teach William as he was often away in England pursuing legal business. Archibald Hamilton had not had a university education and it is thought that Hamilton's genius came from his mother, Sarah Hutton. By the age of five, William had already learned Latin, Greek, and Hebrew. He was taught these subjects by his uncle, the Rev James Hamilton, who William lived with in Trim for many years. James was a fine teacher. William soon mastered additional languages but a turning point came in his life at the age of 12 when he met the American Zerah Colburn. Colburn could perform amazing mental arithmetical feats and Hamilton joined in competitions of arithmetical ability with him. It appears that losing to Colburn sparked Hamilton's interest in mathematics. Hamilton's introduction to mathematics came at the age of 13 when he studied Clairaut's Algebra, a task made somewhat easier as Hamilton was fluent in French by this time. At age 15 he started studying the works of Newton and Laplace. In 1822 Hamilton found an error in Laplace's Méchanique céleste and, as a result of this, he came to the attention of John Brinkley, the Astronomer Royal of Ireland, who said: *This young man, I do not say will be, but is, the first mathematician of his age.*

Hamilton entered Trinity College, Dublin at the age of 18 and in his first year he obtained an 'optime' in Classics, a distinction only awarded once in 20 years. He achieved this merit despite spending most of his time living with his Cousin Arthur at Trim and therefore not attending all of his lectures. In August 1824, Uncle James took Hamilton to Summerhill to meet the Disney family. It was at this point that William first met their daughter Catherine and immediately fell hopelessly in love with her. Unfortunately, as he had three years left at Trinity College, Hamilton was not in a position to propose marriage. However Hamilton was making remarkable progress for an undergraduate and submitted his first paper to the Royal Irish Academy before the end of 1824, which was entitled On Caustics. The following February, Catherine's mother informed William that her daughter was to marry a clergyman, who was fifteen years her senior. He was affluent and could offer more to Catherine than Hamilton. In his next set of exams William was given a 'bene' instead of the usual 'valde bene' due to the fact that he was so distraught at losing Catherine. He became ill and at one point he even considered suicide. In this period he turned to poetry, which was a habit that he pursued for the rest of his life in times of anguish.

In 1826 Hamilton received an 'optime' in both science and Classics, which was unheard of, while in his final year as an undergraduate he presented a memoir Theory of Systems of Rays to the Royal Irish Academy. It is in this paper that Hamilton introduced the characteristic function for optics. Hamilton's finals examiner, Boyton, persuaded him to apply for the post of Astronomer Royal at Dunsink observatory even although there had already been six applicants, one of whom was George Biddell Airy. Later in 1827 the board appointed Hamilton Professor of Astronomy at Trinity College while he was still an undergraduate aged twenty-one years. This appointment brought a great deal of controversy as Hamilton did not have much experience in observing. His predecessor, Professor Brinkley, who had become a bishop, did not think that it had been the correct decision for Hamilton to accept the post and implied that it would have been prudent for him to have waited for a fellowship. It turned out that Hamilton had made an poor choice as he lost interest in astronomy and spend all time on mathematics.

Before beginning his duties in this prestigious position, Hamilton toured England and Scotland (from where the Hamilton family originated). He met the poet Wordsworth and they became friends. One of Hamilton's sisters Eliza wrote poetry too and when Wordsworth came to Dunsink to visit, it was her poems that he liked rather than Hamilton's. The two men had long debates over science versus poetry. Hamilton liked to compare the two, suggesting that mathematical language was as artistic as poetry. However, Wordsworth disagreed saying that: *Science applied only to material uses of life waged war with and wished to extinguish imagination.* Wordsworth had to tell Hamilton quite forcibly that his talents were in science rather than poetry: *You send me showers of verses which I receive with much pleasure ... yet have we fears that this employment may seduce you from the path of science. ... Again I do venture to submit to your consideration, whether the poetical parts of your nature would not find a field more favourable to their nature in the regions of prose, not because those regions are humbler, but because they may be gracefully and profitably trod, with footsteps less careful and in measures less elaborate.*

Hamilton took on a pupil by the name of Adare. They were a bad influence on each other as Adare's eyesight started to present problems as he was doing too much observing, while at the same time Hamilton became ill due to overwork. They decided to take a trip to Armagh by way of a holiday and visit another astronomer Romney Robinson. It was on this occasion that Hamilton met Lady Campbell, who was to become one of his favourite confidants. William also took the opportunity to visit Catherine, as she was living relatively nearby, which she then reciprocated by coming to the observatory. Hamilton was so nervous in her presence that he broke the eyepiece of the telescope whilst trying to give her a demonstration. This episode inspired another interval of misery and poem writing. In July 1830 Hamilton and his sister Eliza visited Wordsworth and it was around this time that he started to think seriously about getting married. He considered Ellen de Vere, and he told Wordsworth that he: ... *admired her mind* ... but he did not mention love. He did, however, bombard her with poetry and was about to propose marriage when she happened to say that she could ... *not live happily anywhere but at Curragh.* Hamilton thought this was her way of discouraging him tactfully and so he ceased to pursue her. However he was proved to be mistaken as she married the following year and did leave Curragh! Fortunately, one good thing transpired from the event as Hamilton became firm friends with Ellen's brother Aubrey although a dispute about religion in 1851 made them go their separate ways.

Catherine aside, Hamilton seemed quite fickle when it came to relationships with women. Perhaps this was because he thought that he ought to marry and so, if he could not have Catherine, then it did not really matter who he married. In the end he married Helen Maria Bayly who lived just across the fields from the observatory. William told Aubrey that she was "not at all brilliant" and, unfortunately, the marriage was fated from the start. They spent their honeymoon at Bayly Farm and Hamilton worked on his third supplement to his Theory of Systems of Rays for the duration. Then at the observatory Helen did not have much of an idea of housekeeping and was so often ill that the household became extremely disorganised. In the years to come she spent most of her time away from the observatory as she was looking after her ailing mother or was indisposed herself.

In 1832 Hamilton published this third supplement to Theory of Systems of Rays which is essentially a treatise on the characteristic function applied to optics. Near the end of the work he applied the characteristic function to study Fresnel's wave surface. From this he predicted conical refraction and asked the Professor of Physics at Trinity College, Humphrey Lloyd, to try to verify his theoretical prediction experimentally. This Lloyd did two months later and this theoretical prediction brought great fame to Hamilton. However, it also led to controversy with MacCullagh, who had come very close to the theoretical discovery himself but, he was forced to admit, had failed to take the last step.

On 4 November 1833 Hamilton read a paper to the Royal Irish Academy expressing complex numbers as algebraic couples, or ordered pairs of real numbers. He used algebra in treating dynamics in On a General Method in Dynamics in 1834. In this paper Hamilton gave his first statement of the characteristic function applied to dynamics and wrote a second paper on the topic the following year. Hankins writes : *These papers are difficult to read. Hamilton presented his arguments with great economy, as usual, and his approach was entirely different from that now commonly presented in textbooks describing the method. In the two essays on dynamics Hamilton first applied the characteristic function V to dynamics just as he had in optics, the characteristic function being the action of the system in moving from its initial to its final point in configuration space. By his law of varying action he made the initial and final coordinates the independent variables of the characteristic function. For conservative systems, the total energy H was constant along any real path but varied if the initial and final points were varied, and so the characteristic function in dynamics became a function of the 6n coordinates of initial and final position (for n particles) and the Hamiltonian H.*

The year 1834 was the one in which Hamilton and Helen had a son, William Edwin. Helen then left Dunsink for nine months leaving Hamilton to fight the loneliness by throwing himself into his work even more. In 1835 Hamilton published Algebra as the Science of Pure Time which were inspired by his study of Kant and presented to a meeting of the British Association for the Advancement of Science. This second paper on algebraic couples identified them with steps in time and he referred to the couples as 'time steps'.

Hamilton was knighted in 1835 and that year his second son, Archibald Henry, was born but the next few years did not bring him much happiness. After the discovery of algebraic couples, he tried to extend the theory to triplets, and this became an obsession that plagued him for many years. The following autumn he went to Bristol for a meeting of the British Association, and Helen took the children with her to Bayly Farm for ten months. His cousin Arthur died, and not long after Helen returned from her mother's she went away again to England this time leaving the children behind after the birth of a daughter, Helen Eliza Amelia. At this point, William became depressed and started to have problems with alcohol so his sister came back to live at Dunsink. Helen returned in 1842 when Hamilton was so preoccupied with the triplets that even his children were aware of it. Every morning they would inquire: *Well, Papa can you multiply triplets?* but he had to admit that he could still only add and subtract them.

On 16 October 1843 (a Monday) Hamilton was walking in along the Royal Canal with his wife to preside at a Council meeting of the Royal Irish Academy. Although his wife talked to him now and again Hamilton hardly heard, for the discovery of the quaternions, the first noncommutative algebra to be studied, was taking shape in his mind: *And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples ... An electric circuit seemed to close, and a spark flashed forth.* He could not resist the impulse to carve the formulae for the quaternions $i^2=j^2=k^2=ijk=-1$. in the stone of Brougham Bridge as he and his wife passed it. Hamilton felt this discovery would revolutionise mathematical physics and he spent the rest of his life working on quaternions. He wrote: *I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions* [*the calculus*] *was for the close of the seventeenth.*

Shortly after Hamilton's discovery of the quaternions his personal life started to prey on his mind again. In 1845, Thomas Disney visited Hamilton at the observatory and brought Catherine with him. This must have upset William as his alcohol dependency took a turn for the worse. At a meeting of the Geological Society the following February he made an exhibition of himself through his intoxication. Macfarlane writes: *... at a dinner of a scientific society in Dublin he lost control of himself, and was so mortified that, on the advice of friends he resolved to abstain totally. This resolution he kept for two years, when ... he was taunted for sticking to water, particularly by Airy ... . He broke his good resolution, and from that time forward the craving for alcoholic stimulants clung to him.*

The year 1847 brought the deaths of his uncles James and Willey and the suicide of his colleague at Trinity College, James MacCullagh, which greatly disturbed him despite the fact that they had not always seen eye to eye. The following year Catherine began writing to Hamilton, which cannot have helped at this time of depression. The correspondence continued for six weeks and became more informal and personal until Catherine felt so guilty that she confessed to her husband. Hamilton wrote to Barlow and informed him that they would never hear from him again. However, Catherine wrote once more and this time attempted suicide (unsuccessfully) as her remorse was so great. She then spent the rest of her life living with her mother or siblings, although there was no official separation from Barlow. Hamilton persisted in his correspondence to Catherine, which he sent through her relatives. It is no surprise that Hamilton gave in to alcohol immediately after this, but he threw himself into his work and began writing his Lectures on Quaternions. He published Lectures on Quaternions in 1853 but he soon realised that it was not a good book from which to learn the theory of quaternions. Perhaps Hamilton's lack of skill as a teacher showed up in this work. Hamilton helped Catherine's son James to prepare for his Fellowship examinations which were on quaternions. He saw this as revenge towards Barlow as he was able to help his son in a way that his father could not. Later that year Hamilton received a pencil case from Catherine with an inscription that read : *From one who you must never forget, nor think unkindly of, and who would have died more contented if we had once more met.* Hamilton went straight to Catherine and gave her a copy of Lectures on Quaternions. She died two weeks later. As a way of dealing with his grief, Hamilton plagued the Disney family with incessant correspondence, sometimes writing two letters a day. Lady Campbell was another sufferer of the burden of mail, as only she and the Disneys knew of his love for Catherine. On the other hand, Helen must have always suspected that she did not take first place in her husband's heart, a notion that must have been strengthened in 1855 when she found a letter from Dora Disney (Catherine's sister-in-law). This led to an argument, although the only consequence was that Dora had her letters addressed by her husband, they did not stop altogether.

Determined to produce a work of lasting quality, Hamilton began to write another book Elements of Quaternions which he estimated would be 400 pages long and take 2 years to write. The title suggests that Hamilton modelled his work on Euclid's Elements and indeed this was the case. The book ended up double its intended length and took seven years to write. In fact the final chapter was incomplete when he died and the book was finally published with a preface by his son William Edwin Hamilton. Not everyone found Hamilton's quaternions the answer to everything they had been looking for. Thomson wrote: *Quaternions came from Hamilton after his really good work had been done, and though beautifully ingenious, have been an unmixed evil to those who have touched them in any way.* Cayley compared the quaternions with a pocket map : ... *which contained everything but had to be unfolded into another form before it could be understood.*

Hamilton died from a severe attack of gout shortly after receiving the news that he had been elected the first foreign member of the National Academy of Sciences of the USA.