Basic Algebra

6. Homomorphisms of rings

Ferdinand Georg Frobenius†
(1849-1917)

Joseph Henry Maclagen Wedderburn††
(1882-1948)

6.1. Let X be a set ane let $\mathfrak{P}(X)$ be the power set ring of X. The map $e : \mathfrak{P}(X) \to (\mathrm{K}_2)^X$, that assigns every subset $A \subseteq X$ to its characteristic funktion e_A (which maps $x \mapsto 1$ if $x \in A$ and $x \mapsto 0$ if $x \notin A$), is an isomorphism of rings of the ring $\mathfrak{P}(X)$ onto the X–fold direct product of K_2. A permutation σ of X induces the permutation σ' of $\mathfrak{P}(X)$ with $\sigma'(A) := \sigma(A) = \{\sigma(x) : x \in A\}$ for $A \in \mathfrak{P}(X)$. For $\sigma \in \mathfrak{S}(X)$, $\sigma' \in \operatorname{Aut}\mathfrak{P}(X)$, and the map $\sigma \mapsto \sigma'$ is an isomorphism of groups from the group $\mathfrak{S}(X)$ onto $\operatorname{Aut}\mathfrak{P}(X)$.

6.2. Let $\varphi : A \to A'$ be a homomorphism of rings. For a ring B denote $\Psi_B(\varphi) : \operatorname{Hom}(B, A) \to \operatorname{Hom}(B, A')$ and $\Phi_B(\varphi) : \operatorname{Hom}(A', B) \to \operatorname{Hom}(A, B)$ the cannonical maps (on the sets of ring homomorphisms) with $\tau \mapsto \varphi\tau$ resp. $\sigma \mapsto \sigma\varphi$.

a). φ is injective if and only if $\Psi_B(\varphi)$ is injective for all rings B. (**Hint:** Construct the ring $\mathbb{Z} \times (\operatorname{Kern}\varphi)$ and consider the ring homomorphism $\mathbb{Z} \times (\operatorname{Kern}\varphi) \to A$ defined by $(n, a) \mapsto n \cdot 1_A + a$ and $(n, a) \mapsto n \cdot 1_A$.)

b). If φ is surjective, then $\Phi_B(\varphi)$ is injective for all rings B. (**Remark:** The converse does not hold: For every ring B, $\Phi_B(\mathbb{Z} \to \mathbb{Q})$ is injective, but the inclusion $\mathbb{Z} \to \mathbb{Q}$ is not surjective.)

6.3. Let $m, n \in \mathbb{N}^*$ and n be a divisor of m. Then there exists exactly one ring homomorphism $\varphi : \mathrm{A}_m \to \mathrm{A}_n$. Further, both φ as well as $\varphi^\times : \mathrm{A}_m^\times \to \mathrm{A}_n^\times$ are surjective. (**Hint:** If $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$, then there exists a $r \in \mathbb{N}$ such that $\gcd(a + rn, m) = 1$, for example, the product of those prime factors of m, which do not divide a or n.)

6.4. a). Let $n \in \mathbb{N}$. Show that the projections $p_i : (a_i) \mapsto a_i$, $i = 1, \ldots, n$, are the only ring homomorphisms $\mathbb{Z}^n \to \mathbb{Z}$. (**Hint:** Consider $e_r e_s = 0$ for $r \neq s$, where e_r, $r = 1, \ldots, n$, is the standard $\mathbb{Z}$–basis of $\mathbb{Z}^n$.)

b). Show that the projections $p_i : (a_j)_{j\in\mathbb{N}} \mapsto a_i$, $i \in \mathbb{N}$, are the only ring homomorphisms $\mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$. (**Hint:** Use the following theorem:

Theorem (Specker) *The projections $\pi_m : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$, $m \in \mathbb{N}$, with $(a_n)_{n\in\mathbb{N}} \mapsto a_m$ form a basis of the $\mathbb{Z}$–module of all linear forms on $\mathbb{Z}^{\mathbb{N}}$.*

Corollary (R. Baer, E. Specker) *Let I be an infinite set. Then the ablelian group $\mathbb{Z}^I$ is not a free group. In particular, $\mathbb{Z}^{\mathbb{N}}$ is not free.*

Proof Suppose that $\mathbb{Z}^{\mathbb{N}}$ is free with the basis f_i, $i \in I$, then I will be necessarily uncountable. But then there will be uncountably many coordinate functions f_i^*, $i \in I$ and hence there will be uncountably many linear forms on $\mathbb{Z}^{\mathbb{N}}$, but by the theorem there are only countably many linear forms on $\mathbb{Z}^{\mathbb{N}}$. •

Proof of the theorem: (Due to E. Specker). First we shall show that π_m, $m \in \mathbb{N}$, are linearly independent over $\mathbb{Z}$. Let e_n, $n \in \mathbb{N}$, be the standard basis of $\mathbb{Z}^{(\mathbb{N})} \subseteq \mathbb{Z}^{\mathbb{N}}$. Then $\pi_m(e_n) = \delta_{mn}$ and from $\sum_{m\in\mathbb{N}} a_m\pi_m = 0$, $a_m \in \mathbb{Z}$, it follows that $0 = \sum_{m\in\mathbb{N}} a_m\pi_m(e_n) = \sum_{m\in\mathbb{N}} a_m\delta_{mn} = a_n$ for all $n \in \mathbb{N}$.

Now it remain to show that every linear form on $\mathbb{Z}^{\mathbb{N}}$ is a linear combination of the π_m, $m \in \mathbb{N}$. Let $h : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ be a given linear form and let $b_n := h(e_n)$. Let c_n, $n \in \mathbb{N}$, be a sequence of positive natural numbers such that c_{n+1} is a multiple of c_n for all $n \in \mathbb{N}$ and such that $c_{n+1} \geq n + 1 + \sum_{r=0}^{n} |c_r b_r|$ for all

$n \in \mathbb{N}$. Such a sequence can be defined recursively. We consider $c := h((c_n)_{n\in\mathbb{N}})$. For every $m \in \mathbb{N}$, there exists a $y_m \in \mathbb{Z}^{\mathbb{N}}$ such that $(c_n) = \sum_{n=0}^{m} c_n e_n + c_{m+1} y_m$. Applying h we get $c = \sum_{n=0}^{m} c_n b_n + c_{m+1} h(y_m)$. and so $|c - \sum_{n=0}^{m} c_n b_n| = c_{m+1}|h(y_m)|$ which is either 0 or $\geq c_{m+1}$. If $m \geq |c|$, then $|c - \sum_{n=0}^{m} c_n b_n| \leq |c| + \sum_{n=0}^{m} |c_n b_n| \leq m + \sum_{n=0}^{m} |c_n b_n| < c_{m+1}$ by definition of c_n. Therefore $c = \sum_{n=0}^{m} c_n b_n$ frr all $m \geq |c|$. This means that $c_n b_n = 0$ and so $b_n = 0$ for all $n > |c|$. Therefore the linear form $h - \sum_{n=0}^{|c|} b_n \pi_n$ vanishes on all elements of the standard basis $e_n,\ n \in \mathbb{N}$. We shall now show that such a linear from must be zero.

Therefore let g be a linear form on $\mathbb{Z}^{\mathbb{N}}$ with $g(e_n) = 0$ for all $n \in \mathbb{N}$. Let $(c_n) \in \mathbb{Z}^{\mathbb{N}}$ be given. There exists integers v_n, w_n such that $v_n 2^n + w_n 3^n = c_n$. (for example $c_n = c_n(3-2)^{2n}$.) Then $g((c_n)) = g((v_n 2^n)) + g((w_n 3^n))$. For every $m \in \mathbb{N}$ there exists a $z_m \in \mathbb{Z}^{\mathbb{N}}$ with $(v_n 2^n) = \sum_{n=0}^{m-1} v_n 2^n e_n + 2^m z_m$. It follows that $g((v_n 2^n)) \in 2^m\mathbb{Z}$ for all $m \in \mathbb{N}$, and so $g((v_n 2^n)) = 0$. Analogously it follows that $g((w_n 3^n)) = 0$. This proves that $g((c_n)) = 0$, as desired. •

c). What are all the automorphisms of the ring $\mathbb{Z}^n,\ n \in \mathbb{N}$, resp. $\mathbb{Z}^{\mathbb{N}}$?

6.5. (Radical of an ideal) Let $\mathfrak{a}$ be an ideal in a commutative ring A. The inverse image of the nilradical of $A/\mathfrak{a}$ in A under the canonical projection $A \to A/\mathfrak{a}$ is called the radical of $\mathfrak{a}$. It is usually denoted by $\mathfrak{r}(\mathfrak{a})$ or by $\sqrt{\mathfrak{a}}$. $\mathfrak{r}(\mathfrak{a})$ is the set of all $a \in A$, for which there is (dependent on a) $n \in \mathbb{N}$ such that $a^n \in \mathfrak{a}$. The residue ring $A/\mathfrak{r}(\mathfrak{a})$ is canonically isomorphic to the reduction of $A/\mathfrak{a}$. Further, $\mathfrak{r}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\mathfrak{a})$.

6.6. (Theorem of M. H. Stone) Let A be a Boolean ring. Let $\mathfrak{M}$ denote the set of all maximal ideals in A. Then $\bigcap_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m} = 0$. For every $\mathfrak{m} \in \mathfrak{M}$ there exists a unique homomorphism $\varphi_{\mathfrak{m}} : A \to \mathrm{K}_2$ with the kernel $\mathfrak{m}$. The map $\varphi : A \to \mathrm{K}_2^{\mathfrak{M}}$ defined by $a \mapsto (\varphi_{\mathfrak{m}}(a))_{\mathfrak{m}\in\mathfrak{M}}$ is an injective ring homomorphism. If A has only finitely many elements, then φ is bijective. (**Hint:** Distinct maximal ideals are relatively coprime.)

From the theorem of Stone deduce that: Every Boolean ring is isomorphic to a subring of a full power set ring; every finite Boolean ring is isomorphic to a full power set ring. (see also exercise 6.1) There exists a Boolean ring, which is not isomorphic to the full power set ring. (**Hint:** Full power set ring is never countablly infinite.)

6.7. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_m$ be two-sided ideals in a ring A with $\mathfrak{a}_i + \mathfrak{b}_j = A$ for all i, j. Then $\mathfrak{a}_1 \cdots \mathfrak{a}_n + \mathfrak{b}_1 \cdots \mathfrak{b}_m = A$. Deduce that: If $\mathfrak{a}$ and $\mathfrak{b}$ are relatively coprime two-sided ideals in a ring, then $\mathfrak{a}^n$ and $\mathfrak{b}^m$ are also relatively coprime ideals for arbitrary $m, n \in \mathbb{N}$.

6.8. For pairwise relatively coprime two-sided ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in a ring A we have :

$$\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = \sum_{\sigma\in\mathfrak{S}_n} \mathfrak{a}_{\sigma(1)} \cdots \mathfrak{a}_{\sigma(n)}\,. \qquad \textbf{(Hint: } \text{Induction)}$$

If A is commutative, then $\bigcap_{i=1}^{n} \mathfrak{a}_i = \prod_{i=1}^{n} \mathfrak{a}_i$.

6.9. Let A be a ring. The set $\mathrm{Idp}(\mathrm{Z}(A))$ of all idempotent elements in the center $\mathrm{Z}(A)$ of A is finite if and only if A is isomorphic to a finite direct product of indecomposable rings. Moreover, in this case $|\mathrm{Idp}(\mathrm{Z}(A))| = 2^s$, where s is the number of indecomposable components in the product representation of A.

6.10. Let $\varphi : A \to B$ be a homomorphism of commutative rings. φ induces a map $\varphi_1 : \mathrm{Idp}(A) \to \mathrm{Idp}(B)$ between the idempotent elements of A resp. B. If φ is surjective and the kernel of φ is conatined in the nilradical $\mathfrak{n}_A$ of A, then φ_1 is bijective. (**Hint:** Use exercise 1.4) **Corollary:** A commutative ring A is indecomposable (see T6.8) if and only if $A/\mathfrak{n}_A$ is indecomposable. (**Hint:** If $A/\mathfrak{n}_A$ is indecomposable, then so is A even in the non-commutative case (proof!). However, in the non-commutative case the residue ring $A/\mathfrak{n}_A$can be decomposable without A being so. For example the ring

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a\,,\ b\,,\ c \in K \right\} \subseteq \mathrm{M}_2(K)$$

of 2×2 upper triangular matrices over a field K (vgl. Kapitel V) is indecomposable, but the residue ring $B/\mathfrak{n}_B \cong K \times K$ is decomposable.)

Below one can see (simple) test-exercises.

Test-Exercises

T6.1. Let A and B be rings and let f, g be homomorphisms of A in B. The subset A' of all elements $a \in A$ with $f(a) = g(a)$ is a subring of A. Moreover, if A is a division ring, then A' is also a division ring.

T6.2. Let K and L be fields of characteristic $\neq 2$ and φ be a group homomorphism of $(K, +)$ in $(L, +)$ with the following properties: (1) $\varphi(1) = 1$. (2) $\varphi(a)\varphi(a^{-1}) = 1$ for all $a \in K$, $a \neq 0$. Show that φ is a ring homomorphism of K in L. (**Hint:** If $a \in K$, $a \neq 0$, $a \neq 1$, then prove $\varphi(a^2) = \varphi(a)^2$ by considering $1 + a = (1 - a)^{-1} - ((a^2)^{-1} - a^{-1})^{-1}$. — **Remark:** More generally, (Satz von Hua) if K and L are division rings and $\varphi : K \to L$ is a homomorphism of its addtive groups with the properties (1) and (2), then φ is a ring homomorphism or a ring anti–homomorphism.)

T6.3. Two subrings of $\mathbb{Q}$ are isomorphic if and only if they are equal. (**Remark:** The set of isomorphism classes of subrings of $\mathbb{Q}$ has the cardinality of the continuum, i.e., $\mathrm{c} := \mathrm{card}(\mathbb{R})$.)

T6.4. Let A be an ordered ring.

a). For a natural number $n \in \mathbb{N}^*$, show that the equation $x^n = 1$ has atmost two solutions 1 and -1 in A. (**Hint:** One may assume $x \geq 0$.)

b). Let $a, b \in A$. Show that $|a| \cdot |b| = |ab|$, $|a|^2 = a^2$, $|a + b| \leq |a| + |b|$, $|a - b| \geq ||a| - |b||$.

c). If the order on A is archimedian, then show that the identity map is the only order preserving automorphism of A.

T6.5. (Reduced rings) Let A be a commutative ring. Then A is called reduced, if zero is the only nilpotent element of A. A ring A is reduced if and only if the nilradical $\mathfrak{n}_A$ of A is the zero ideal. The residue ring $A/\mathfrak{n}_A$ is reduced. (This ring is called the reduction of A and is usually denoted by A_{red}.)

T6.6. Let A be a ring which contain the prime field $\mathbb{Q}$ of characteristic 0 and as an $\mathbb{Q}$–algebra generated by a nilpotent element $a \in A$. Then A is commutative, Aa is the additive group of the nilpotent elements of A and $1 + Aa$ is the multiplicative group of unipotent elements of A. The exponential map

$$\exp : c \mapsto \sum_{n=0}^{\infty} \frac{c^n}{n!}$$

of Aa into $1 + Aa$ is an isomorphism of groups. (Prove its bijectivity by induction on the number m, for which $a^m = 0$, where the induction hypothesis is applied on A/Aa^{m-1}.)

T6.7. Let A be a ring and let I be a finite indexed set. The set of the family $(e_i)_{i \in I}$ of idempotent elements $e_i \in A$ such that $e_i \in \mathrm{Z}(A)$, $e_i e_j = \delta_{ij} e_i$, $\sum_{i \in I} e_i = 1$ defines

$$(e_i)_{i \in I} \mapsto (\mathfrak{a}_i)_{i \in I} \quad \text{with} \quad \mathfrak{a}_i := \sum_{j \neq i} Ae_j = A(1 - e_i)$$

a bijective map onto the set of the family $(\mathfrak{a}_i)_{i \in I}$ of two-sided ideals $\mathfrak{a}_i$ in A, which satisfy the conditions $\mathfrak{a}_i + \mathfrak{a}_j = A$ for all $i, j \in I$ with $i \neq j$ and $\bigcap_{i \in I} \mathfrak{a}_i = 0$.

T6.8. A non-zero ring which is not isomorphic to a product of two non-zero rings is called indecomposable or connected.

Let A be a non-zero ring. The following statements are equivalent:

(1) A is indecomposable.

(2) There are no relatively coprime two-sided non-unit ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A such that $\mathfrak{a} \cap \mathfrak{b} = 0$.

(3) The center $\mathrm{Z}(A)$ of A is indecomposable.

(4) There are no idepotents other than 0 and 1 in $\mathrm{Z}(A)$.

(**Hint:** Use the exercise and for (4): If e is an idempotent, then so is $1 - e$ and $1 = e + (1 - e)$, $e(1 - e) = 0$.)

T6.9. The characteristic of an indecomposable ring is 0 or a power of a prime number.

T6.10. The number of elements in a finite indecomposable ring is a power of a prime number.

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† **Ferdinand Georg Frobenius (1849-1917)** was born on 26 Oct 1849 in Berlin-Charlottenburg, Prussia (now Germany) and died on 3 Aug 1917 in Berlin, Germany. Georg Frobenius's father was Christian Ferdinand Frobenius,

a Protestant parson, and his mother was Christine Elizabeth Friedrich. Georg was born in Charlottenburg which was a district of Berlin which was not incorporated into the city until 1920. He entered the Joachimsthal Gymnasium in 1860 when he was nearly eleven years old and graduated from the school in 1867. In this same year he went to the University of Göttingen where he began his university studies but he only studied there for one semester before returning to Berlin.

Back at the University of Berlin he attended lectures by Kronecker, Kummer and Weierstrass. He continued to study there for his doctorate, attending the seminars of Kummer and Weierstrass, and he received his doctorate (awarded with distinction) in 1870 supervised by Weierstrass. In 1874, after having taught at secondary school level first at the Joachimsthal Gymnasium then at the Sophienrealschule, he was appointed to the University of Berlin as an extraordinary professor of mathematics.

For the description of Frobenius's career so far, the attentive reader may have noticed that no mention has been made of him receiving an habilitation before being appointed to a teaching position. This is not an omission, rather it is surprising given the strictness of the German system that this was allowed. We should say that it must ultimately have been made possible due to strong support from Weierstrass who was extremely influential and considered Frobenius one of his most gifted students.

Frobenius was only in Berlin for a year before he went to Zürich to take up an appointment as an ordinary professor at the Eidgenössische Polytechnikum. For seventeen years, between 1875 and 1892, Frobenius worked in Zürich. He married there and brought up a family and did much important work in widely differing areas of mathematics. We shall discuss some of the topics which he worked on below, but for the moment we shall continue to describe how Frobenius's career developed.

In the last days of December 1891 Kronecker died and, therefore, his chair in Berlin became vacant. Weierstrass, strongly believing that Frobenius was the right person to keep Berlin in the forefront of mathematics, used his considerable influence to have Frobenius appointed. However, for reasons which we shall discuss in a moment, Frobenius turned out to be something of a mixed blessing for mathematics at the University of Berlin.

The positive side of his appointment was undoubtedly his remarkable contributions to the representation theory of groups, in particular his development of character theory, and his position as one of the leading mathematicians of his day. The negative side came about largely through his personality which is described as: ... occasionally choleric, quarrelsome, and given to invectives.

Biermann, looks more closely at his character (no pun intended!), and how it affected the success of mathematical education at the university. He describes the strained relationships which developed between Frobenius and his colleagues at Berlin. He had such high standards that in the end these did not serve Berlin well. He suspected at every opportunity a tendency of the Ministry to lower the standards at the University of Berlin, in the words of Frobenius, to the rank of a technical school ... Even so, Fuchs and Schwarz yielded to him, and later Schottky, who was indebted to him alone for his call to Berlin. Frobenius was the leading figure, on whom the fortunes of mathematics at Berlin university rested for 25 years. Of course, it did not escape him, that the number of doctorates, habilitations, and docents slowly but surely fell off, although the number of students increased considerably. That he could not prevent this, that he could not reach his goal of maintaining unchanged the times of Weierstrass, Kummer and Kronecker also in their external appearances, but to witness helplessly these developments, was doubly intolerable for him, with his choleric disposition.

We should not be too hard on Frobenius for, as Haubrich explains, Frobenius's attitude was one which was typical of all professors of mathematics at Berlin at this time: They all felt deeply obliged to carry on the Prussian neo-humanistic tradition of university research and teaching as they themselves had experienced it as students. This is especially true of Frobenius. He considered himself to be a scholar whose duty it was to contribute to the knowledge of pure mathematics. Applied mathematics, in his opinion, belonged to the technical colleges.

The view of mathematics at the University of Göttingen was, however, very different. This was a time when there was competition between mathematians in the University of Berlin and in the University of Göttingen, but it was a competition that Göttingen won, for there mathematics flourished under Klein, much to Frobenius's annoyance. Biermann writes that: *The aversion of Frobenius to Klein and S Lie knew no limits ...*

Frobenius hated the style of mathematics which Göttingen represented. It was a new approach which represented a marked change from the traditional style of German universities. Frobenius, as we said above, had extremely traditional views. In a letter to Hurwitz, who was a product of the Göttingen system, he wrote on 3 February 1896: *If you were emerging from a school, in which one amuses oneself more with rosy images than hard ideas, and if, to my joy, you are also gradually becoming emancipated from that, then old loves don't rust. Please take this joke facetiously.*

One should put the other side of the picture, however, for Siegel, who knew Frobenius for two years from 1915 when he became a student until Frobenius's death, relates his impression of Frobenius as having a warm personality and expresses his appreciation of his fast-paced varied and deep lectures. Others would describe his lectures as solid but not stimulating. To gain an impression of the quality of Frobenius's work before the time of his appointment to Berlin in 1892 we can do no better than to examine the recommendations of Weierstrass and Fuchs when Frobenius was elected to the Prussian Academy of Science in 1892. Fairly extensive quotes from this document, and another similar document from Fuchs and Helmholtz, are given, but we quote a short extract to show the power, variety and high quality of Frobenius's work in his Zürich years. Weierstrass and Fuchs list 15 topics on which Frobenius had made major contributions:

1.On the development of analytic functions in series. 2.On the algebraic solution of equations, whose coefficients are rational functions of one variable. 3.The theory of linear differential equations. 4.On Pfaff's problem. 5.Linear forms with integer coefficients. 6.On linear substitutions and bilinear forms... 7.On adjoint linear differential operators... 8.The theory of elliptic and Jacobi functions... 9.On the relations among the 28 double tangents to a plane of degree 4. 10.On Sylow's theorem. 11.On double cosets arising from two finite groups. 12.On Jacobi's covariants... 13.On Jacobi functions in three variables. 14.The theory of biquadratic forms. 15.On the theory of surfaces with a differential parameter.

In his work in group theory, Frobenius combined results from the theory of algebraic equations, geometry, and number theory, which led him to the study of abstract groups. He published über Gruppen von vertauschbaren Elementen in 1879 (jointly with Stickelberger, a colleague at Zürich) which looks at permutable elements in groups. This paper also gives a proof of the structure theorem for finitely generated abelian groups. In 1884 he published his next paper on finite groups in which he proved Sylow's theorems for abstract groups (Sylow had proved theorem as a result about permutation groups in his original paper). The proof which Frobenius gives is the one, based on conjugacy classes, still used today in most undergraduate courses.

In his next paper in 1887 Frobenius continued his investigation of conjugacy classes in groups which would prove important in his later work on characters. In the introduction to this paper he explains how he became interested in abstract groups, and this was through a study of one of Kronecker's papers. It was in the year 1896, however, when Frobenius was professor at Berlin that his really important work on groups began to appear. In that year he published five papers on group theory and one of them über die Gruppencharactere on group characters is of fundamental importance. He wrote in this paper: *I shall develop the concept [of character for arbitrary finite groups] here in the belief that through its introduction, group theory will be substantially enriched.*

This paper on group characters was presented to the Berlin Academy on July 16 1896 and it contains work which Frobenius had undertaken in the preceding few months. In a series of letters to Dedekind, the first on 12 April 1896, his ideas on group characters quickly developed. Ideas from a paper by Dedekind in 1885 made an important contribution and Frobenius was able to construct a complete set of representations by complex numbers. It is worth noting, however, that although we think today of Frobenius's paper on group characters as a fundamental work on representations of groups, Frobenius in fact introduced group characters in this work without any reference to representations. In was not until the following year that representations of groups began enter the picture, and again it was a concept due to Frobenius. Hence 1897 is the year in which the representation theory of groups was born.

Over the years 1897-1899 Frobenius published two papers on group representations, one on induced characters, and one on tensor product of characters. In 1898 he introduced the notion of induced representations and the Frobenius Reciprocity Theorem. It was a burst of activity which set up the foundations of the whole of the machinery of representation theory.

In a letter to Dedekind on 26 April 1896 Frobenius gave the irreducible characters for the alternating groups A_4, A_5 the symmetric groups S_4, S_5 and the group $\mathrm{PSL}(2,7)$ of order 168. He completely determined the characters of symmetric groups in 1900 and of characters of alternating groups in 1901, publishing definitive papers on each. He continued his applications of character theory in papers of 1900 and 1901 which studied the structure of Frobenius groups.

Only in 1897 did Frobenius learn of Molien's work which he described in a letter to Dedekind as "very beautiful but difficult". He reformulated Molien's work in terms of matrices and then showed that his characters are the traces of the irreducible representations. This work was published in 1897. Frobenius's character theory was used with great effect by Burnside and was beautifully written up in Burnside's 1911 edition of his Theory of Groups of Finite Order.

Frobenius had a number of doctoral students who made important contributions to mathematics. These included Edmund Landau who was awarded his doctorate in 1899, Issai Schur who was awarded his doctorate in 1901, and Robert Remak who was awarded his doctorate in 1910. Frobenius collaborated with Schur in representation theory of groups and character theory of groups. It is certainly to Frobenius's credit that he so quickly spotted the genius of his student Schur. Frobenius's representation theory for finite groups was later to find important applications in quantum mechanics and theoretical physics which may not have entirely pleased the man who had such "pure" views about mathematics.

Among the topics which Frobenius studied towards the end of his career were positive and non-negative matrices. He introduced the concept of irreducibility for matrices and the papers which he wrote containing this theory around 1910 remain today the fundamental results in the discipline. The fact so many of Frobenius's papers read like present day text-books on the topics which he studied is a clear indication of the importance that his work, in many different areas, has had in shaping the mathematics which is studied today. Having said that, it is also true that he made fundamental contributions to fields which had already come into existence and he did not introduce any totally new mathematical areas as some of the greatest mathematicians have done.

Haubrich gives the following overview of Frobenius's work:

The most striking aspect of his mathematical practice is his extraordinary skill at calculations. In fact, Frobenius tried to solve mathematical problems to a large extent by means of a calculative, algebraic approach. Even his analytical work was guided by algebraic and linear algebraic methods. For Frobenius, conceptual argumentation played a somewhat secondary role. Although he argued in a comparatively abstract setting, abstraction was not an end in itself. Its advantages to him seemed to lie primarily in the fact that it can lead to much greater clearness and precision.

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†† **Joseph Henry Maclagen Wedderburn (1882-1948)** was born on 2 Feb 1882 in Forfar, Angus, Scotland and died on 9 Oct 1948 in Princeton, New Jersey, USA. Joseph Wedderburn's father, Alexander Stormonth Maclagen Wedderburn, was a medical doctor. Alexander Wedderburn came from a family of Ministers of the Church with his father being the Parish Minister of Kinfauns and his grandfather (Joseph's great-grandfather) being Parish Minister of Blair Atholl. Joseph's mother was Anne Ogilvie and she came from a family of lawyers; Anne's father had been a lawyer in Dundee. Anne and Alexander Wedderburn had a large family, Joseph being one of fourteen children, eight boys and six girls. In fact Joseph was the tenth child of the family.

Joseph was brought up in Forfar, north of Dundee, and he attended Forfar Academy from the age of five until he was thirteen. He then went to George Watson's College, an independent school in Edinburgh, for three years. In 1898 he completed his school education and won a scholarship to study at the University of Edinburgh. He entered Edinburgh University in 1898, at the age of sixteen and a half. It was a time when Wedderburn made remarkable progress with his mathematics and in addition during 1902-03 he worked as an assistant in the Physical Laboratory of the University. He began mathematical research while still an undergraduate and his first paper, On the isoclinal lines of a differential equation of the first order was published in the Proceedings of The Royal Society of Edinburgh in 1903. Two other papers which he published in the same year in publications of the Royal Society of Edinburgh were on the scalar functions of a vector and on an application of quaternions to differential equations. He obtained an M.A. degree with First Class Honours in mathematics from the University of Edinburgh in 1903.

Wedderburn then pursued postgraduate studies in Germany spending session 1903-1904 at the University of Leipzig and then the summer semester of 1904 at the University of Berlin. Already Wedderburn's mathematical interests were in algebra and his German trip allowed him to interact with Frobenius and Schur. He was awarded a Carnegie Scholarship to study in the United States and he spent 1904-1905 at the University of Chicago where he did joint work with Veblen. Chicago was, of course, an excellent place to continue his deepening interest in algebra for, in addition to Veblen, Eliakim Moore and L E Dickson were there at this time. The determination of finite division algebras was a very natural problem in the light of the work being undertaken in Chicago, and as soon as he arrived at Chicago, Wedderburn started to work on it, in close contact with Dickson.

Returning to Scotland in 1905, Wedderburn worked for four years at the University of Edinburgh as an assistant to George Chrystal. The depth of Wedderburn's contribution to algebra during these years in Edinburgh was remarkable. In 1905 he showed that a non-commutative finite field could not exist. In the paper he published in that year he gave three proofs of this theorem which were all based on a clever use of the interplay between the additive group of a finite division algebra A, and the multiplicative group $A^{*}=A\setminus\{0\}$.

Parshall discusses this theorem. She notes that the first of the three proofs has a gap in it which was not noticed at the time. This is in fact significant since Dickson also found a proof of this result but, since Wedderburn had already found his first "proof" (which Dickson believed to be correct), Dickson acknowledged Wedderburn's priority in a paper he wrote on the topic. Dickson noted in the paper that it is only after having seen his proof that Wedderburn constructed his second and third proofs. Parshall's work here shows that really Dickson should be credited with having found the first correct proof.

This theorem gave, as a corollary, the complete structure of all finite projective geometries. These geometries consisted of a set of "points", a set of "lines" and an "incidence relation" between points and lines, subject only to the conditions that two distinct points are on a single line, two distinct lines have a single common point and a line contains at least three points. Wedderburn and Veblen showed that in all these geometries Pascal's theorem is a consequence of Desargues' theorem. They published the paper Non-Desarguesian and non-Pascalian geometries in the Transactions of the American Mathematical Society in 1907 in which they constructed finite projective geometries which are neither "Desarguesian" nor "Pascalian" (this is Hilbert's terminology).

In 1907 Wedderburn published what is perhaps his most famous paper on the classification of semisimple algebras. In this paper On hypercomplex numbers which appeared in the Proceedings of the London Mathematical Society, he showed that every semisimple algebra is a direct sum of simple algebras and that a simple algebra was a matrix algebra over a division ring. From 1906 to 1908 he served as editor of the Proceedings of the Edinburgh Mathematical Society.

In 1909 Wedderburn returned to the United States being appointed a Preceptor in Mathematics at Princeton where he joined Veblen. We should say a word about the Preceptors at Princeton. They were the idea of Woodrow Wilson (who was to become the 28th President of the United States in 1913). Woodrow Wilson had been Professor of Political Science at Princeton and, in 1902, he was appointed President of Princeton. He set out to change the nature of Princeton by making it a leading research active university. To do this, Wilson said: *... required a large scale infusion of new blood, of scholars who would assume an intimate personal relation with small groups of undergraduates and impart to them something of their own enthusiasm for things of the mind.*

Fifty Preceptors were to be appointed to achieve this in the whole university and Henry Fine, Dean of Mathematics, was put in charge of finding young mathematicians to fill the mathematics posts. Between 1905 and 1909 Eisenhart, Veblen, Bliss, George Birkhoff, and Wedderburn were appointed. The next five years were especially happy ones for Wedderburn and his fellow Preceptors described him during this time:

They recall his passion for play as well as for work, his desire for companionship and association with men. He loved the out-of-doors, found deep satisfaction in the wilderness, in the woods, canoeing along rivers and streams in the company of thoughtful men. As in his scientific work, he brought to the construction of the camp-site, the erection of the tent, the paddling of the canoe up- and down-stream, the qualities of a complete perfectionist. In the wilds of Northern Canada, with congenial men, he found complete happiness. ... His taste in literature ran to books of travel and he accumulated a large library of travel.

However the five happy years came to an end with the outbreak of the First World War. Immediately Wedderburn volunteered for the British Army but, being an exceptionally modest man, he enlisted only in the role of private. Records show that he was the first person at Princeton to volunteer for war service and that he had the longest war service of anyone on the staff. He served in France between January 1918 and March 1919, making use of his scientific skills. In France, as a Captain in the 4th Field Survey Battalion, he devised sound-ranging equipment to pinpoint the positions of enemy guns.

On his return to Princeton he took up his post as Preceptor in Mathematics but he was soon promoted to Assistant Professor in 1920, obtaining permanent tenure as Associate Professor in 1921. He served as Editor of the Annals of Mathematics from 1912 to 1928. From about the end of this period Wedderburn seemed to suffer a mild nervous breakdown and became an increasingly solitary figure. It looks as if from this time on he suffered from depression. Certainly he stopped seeing his friends and although he seemed to recognise that his problems came from loneliness, rather than seek to be with people he deliberately cut himself off. Some of his friends made a strenuous effort to penetrate the barrier he was erecting and found that underneath was still the friendly, deep thinking, brilliant mathematician.

A comment on his teaching by Robert Hooke: *Wedderburn's lecturing style was unique, to say the least. He was apparently a very shy man and much preferred looking at the blackboard to looking at the students. He had the galley proofs from his book "Lectures on Matrices" pasted to cardboard for durability, and his "lecturing" consisted of reading this out loud while simultaneously copying it onto the blackboard. Ernst Snapper, who claimed to be only the fourth person ever with the courage to write a dissertation under Wedderburn (and one of the other three had lost his mind) told me this story explaining why Wedderburn was a bachelor. It seems that an old Scottish tradition required that a man, before marrying, accumulate savings equal to a certain percentage of his annual income. In Wedderburn's case his income had gone up so rapidly that he had never been able to accomplish this.*

By 1945 Princeton gave him early retirement in his own best interests. From this time on his isolation became almost total. Although we have given 9 October 1948 as the date of his death, in fact he probably died a few days earlier than this. The people who looked after the house and grounds in Princeton were he lived found him on that day but the subsequent medical examination revealed that he had died of a heart attack several days earlier.

Parshall writes : *According to officials at the bank which settled Wedderburn's estate ..., the papers remaining at his death were subsequently destroyed, thereby limiting historical study of Wedderburn's life and work almost exclusively to published sources.*

Wedderburn made important advances in the theory of rings, algebras and matrix theory. His best mathematical work was done before his war service and we have referred to some of it above. In total he published around 40 works mostly on rings and matrices. His famous book is Lectures on Matrices (1934). This work was described by Jacobson, who was a student of Wedderburn's. Jacobson writes: *That this was the result of a number of years of painstaking labour is evidenced by the Bibliography of 661 items (in the revised printing) covering the period 1853 to 1936. The work is, however, not a compilation of the literature, but a synthesis that is Wedderburn's own. It contains a number of original contributions to the subject. Though he did not follow the abstract point of view that had just become dominant, neither did he commit the error made by others of treating matrix theory as an art of juggling elements in an array. The important ideas of linear transformations, vector spaces, bilinear forms, though not set off, as is common in most modern treatments, do appear in Wedderburn's book. also, as in his best work, one finds here some neat and suggestive algebraic devices that make the book a very valuable reference book ...*

Among the honours which Wedderburn received were the MacDougall-Brisbane Gold Medal and Prize from the Royal Society of Edinburgh in 1921, and election to the Royal Society of London in 1933.