Basic Algebra

7. Homomorphisms of modules

Richard Dagobert Brauer†
(1901-1977)

Albert Thoralf Skolem††
(1887-1963)

The exercises 7.10, 7.11, 7.15 are marked as $*$ and may be ignored in the first reading.

7.1. Let $\varphi : A \to B$ be a ring homomorphism. If V is a B–module and W is an A– module, there exists a natural group homomorphism (see T7.7 and T7.15 c))

$$\mathrm{Hom}_A(V, W) \to \mathrm{Hom}_B(V, \mathrm{Hom}_A(B, W))$$

defined by $f \mapsto (v \mapsto (b \mapsto f(bv)))$ with inverse $g \mapsto (v \mapsto g(v)(1_B))$. If R is another ring and if V is a B–left–R–right–bimodule, then the above isomorphism is a R–isomorphism. (This holds for example in the case if $R = B$ is commutative.)

7.2. Let A and B be rings. Let U be an A–left-module, V be a B–right-module and W be a (A, B)–bimodule[1]) of Type ${}_AW_B$. Then there exists a natural isomorphism

$$\mathrm{Hom}_A(U, \mathrm{Hom}_B(V, W)) \to \mathrm{Hom}_B(V, \mathrm{Hom}_A(U, W))$$

defined by $f \mapsto (v \mapsto (u \mapsto f(u)(v)))$, with the inverse $g \mapsto (u \mapsto (v \mapsto g(v)(u)))$.

7.3. Let I be a finite set and let A be an indecomposable ring [2])

a). The canonical projections $\beta_i : A^I \to A$, $i \in I$ are the only A–algebra–homomorphisms from $A^I \to A$.

b). The map $\mathfrak{S}(I) \to \mathrm{Aut}_{A\text{–Alg}} A^I$ defined by $\sigma \mapsto ((a_i) \mapsto (a_{\sigma^{-1}(i)}))$ is an isomorphism of groups.

7.4. Let $f : V \to W$ be a homomorphism of modules over a ring A. If $\ker f$ and $\mathrm{im}\, f$ are finite A–modules, then V is also a finite A–module. For the minimal number of generators we have:

$$\mu_A(V) \le \mu_A(\ker f) + \mu_A(\mathrm{im}\, f)\,.$$

7.5. Let K be a division ring, V be a finite dimensional K–vector space, $f : V \to W$ be a linear mapping into an arbitrary K–vector space W and let $U \subseteq V$ be a subspace. Then

$$\mathrm{Dim}_K\, V - \mathrm{Dim}_K\, U \ge \mathrm{Dim}_K\, f(V) - \mathrm{Dim}_K\, f(U)\,.$$

7.6. Let $f : V \to W$, $g : W \to X$ and let $h : X \to Y$ be linear maps of finite dimensional vector spaces over a division ring K.

a). (Inequality of Sylvester) $\mathrm{rank}\, f + \mathrm{rank}\, g - \mathrm{Dim}_K\, W \le \mathrm{rank}(gf) \le \mathrm{Min}\{\mathrm{rank}\, f,\ \mathrm{rank}\, g\}$.
(**Hint :** The first inequality easily follows from the exercise 7.5.)

[1]) See T7.14

[2]) A non-zero ring A is called indecomposable or connected if A is not isomorphic to a product of two non-zero rings. A non-zero ring A is indecomposable if and only if the only idempotents in the center $\mathrm{Z}(A)$ are 0 and 1 (proof!). For example an integral domain is indecomposable.

b). (Inequality of Frobenius) $\operatorname{rank}(hg) + \operatorname{rank}(gf) \le \operatorname{rank} g + \operatorname{rank}(hgf)$. (**Hint:** We may assume that g is surjective and then apply part a).)

7.7. Let V and W be vector spaces over a division ring K. A linear map $f : V \to W$ has finite dimensional kernel if and only if there exists a linear map $g : W \to V$ such that $gf = \mathrm{id}_V + h$, where $h \in \operatorname{End}_K V$ is of finite rank.

7.8. Let V and W be vector spaces over a division ring K. Then

a). The linear maps from V into W of finite rank form a $\mathrm{Z}(K)$–submodule E of $\operatorname{Hom}_K(V, W)$. For $f_1, \ldots, f_n \in E$ and $a_1, \ldots, a_n \in \mathrm{Z}(K)$ we have:

$$\operatorname{rank}(a_1 f_1 + \cdots + a_n f_n) \le \operatorname{rank} f_1 + \cdots + \operatorname{rank} f_n .$$

b). The endomorphisms of V of finite rank form a two-sided ideal in the ring $\operatorname{End}_K V$.

7.9. (Characters) Let M be a (multiplicative) semigroup and let K be a division ring. A non-zero semi-group homomorphism from M in the multiplicative monoid of K is called a character of M with values in K. The constant map $x \mapsto 1_K$ is a character of M, whcih is called the trivial character. If M is a monoid, then every character of M is a monoid homomorphism.[3]) If $a \in K$, $a \ne 0$, then the conjugation $\varkappa_a = (b \mapsto aba^{-1})$ in K is a character of the multiplicative monoid of K with values in K.

a). If $\chi : M \to K$ is a character, where M is finite and $\chi | M^\times$ is not trivial, then $\sum_{x \in M} \chi(x) = 0$. (**Hint:** Let $x_0 \in M^\times$ with $\chi(x_0) \ne 1$. Then $\sum_{x\in M} \chi(x) = \sum_{x \in M} \chi(x_0 x) = \chi(x_0) \sum_{x \in M} \chi(x)$.)

b). (Lemma on characters) Let $\varphi_1, \ldots, \varphi_n$ be characters of M with values in K; Suppose that $\varphi_1, \ldots, \varphi_n$ are (as an elements of K^M) linearly independent over K. If a linear combination $\varphi = \sum_{i=1}^n a_i \varphi_i$ with coefficients $a_i \in K$ is a character of M, then $\varphi = \varkappa_{a_i} \varphi_i$ for every i with $a_i \ne 0$. (**Hint:** For $x, y \in M$ on one side $\varphi(x)\varphi(y) = \sum_i \varphi(x) a_i \varphi_i(y)$, and on the other side $\varphi(x)\varphi(y) = \varphi(xy) = \sum_i a_i \varphi_i(x)\varphi_i(y)$.)

c). (Lemma of Dedekind–Artin) Let K be a field, M be a non-empty semigroup and let φ_i, $i \in I$, be a family of distinct characters of M with values in K. Then φ_i, $i \in I$, are linearly independent over K in K^M. (**Hint:** Use the lemma on characters.)

d). Some applications of the lemma of Dedekind–Artin:

1). Let A, K be algebras over a field k, where A is finite dimensional and K is a field. Then there exist at most $\operatorname{Dim}_k A$ distinct k–algebra–homomorphisms of A in K. (**Hint:** $\operatorname{Hom}_k(A, K)$ is a K–subspace of K^A of the dimension $\operatorname{Dim}_k A$. More generally see T7.16.)

2). Let K be a field. The maps $t \mapsto t^n$, $n \in \mathbb{N}$, are the only polynomial maps of K into itself corresponding characters of the multiplicative monoid of K with values in K. More generally: The functions $t \mapsto t^n$, $n \in \mathbb{Z}$, are the only group homomorphisms of $K^\times \to K^\times$, corresponding to the rational functions on $K^\times$. Deduce that if K is finite, then the multiplicative group $K^\times$ is cyclic. (see also exercise T4.12.)

3). The functions $t \mapsto \exp at$, $a \in \mathbb{C}$, of $\mathbb{R}$ in $\mathbb{C}$ are linearly independent over $\mathbb{C}$.

4). Let K be a field. the sequences $(a^\nu)_{\nu \in \mathbb{N}}$, $a \in K$, are linearly independent over K. (see also exercsie 4.17.)

e). (Inner automorphisms of a division rings) Let K be a division ring with the center k. We consider K as a k–algebra.

1). Let x_i, $i \in I$, be a family of non-zero elements K. Then the inner automorphisms $\varkappa_{x_i}$, $i \in I$, in K^K are linearly independent K if and only if x_i^{-1}, $i \in I$, are linearly independent over k.

(**Hint:** Let $x_0, x_1, \ldots, x_n \in K^\times$ and $x_0^{-1} = \sum_{i=1}^n \alpha_i x_i^{-1}$, $\alpha_i \in k$. Then $x_0 y x_0^{-1} = \sum_{i=1}^n \alpha_i x_0 y x_i^{-1} = \sum_{i=1}^n \alpha_i x_0 x_i^{-1}(x_i y x_i^{-1})$, i.e. $\varkappa_{x_0} = \sum_{i=1}^n a_i \varkappa_{x_i}$, $a_i := \alpha_i x_0 x_i^{-1}$. Conversely, from $\varkappa_{x_0} = \sum_{i=1}^n a_i \varkappa_{x_i}$, all

[3]) *Every homomorphism of a monoid into a monoid in which cancellation holds is a monoid homomorphism, i.e. maps the neutral element onto the neutral element.* Let $\varphi : M \to N$ be a homomorphism. Suppose that cancellation holds in N. Let $e \in M$ and $e' \in N$ be neutral elements. Then $\varphi(e) = \varphi(e^2) = \varphi(e)\varphi(e)$ and on the other hand $\varphi(e) = e'\varphi(e)$. Therefore $\varphi(e) = e'$.

$a_i \neq 0$ and lemma on characters, we get $\varkappa_{x_0} = \varkappa_{a_i}\varkappa_{x_i} = \varkappa_{a_i x_i}$ and therefore $a_i x_i = \alpha_i x_0$ with $\alpha_i \in k$ and $x_0^{-1} = \varkappa_{x_0}(x_0^{-1}) = \sum_{i=1}^n a_i x_i x_0^{-1} x_i^{-1} = \sum_{i=1}^n \alpha_i x_i^{-1}$.)

2). (Theorem of Noether–Skolem–Brauer) If K is finite dimensional over k, then every k–algebra–endomorphism of K is an inner automorphism of K. (**Hint:** Choose a k-basis $y_1^{-1}, \ldots, y_n^{-1}$ of K. Then by 1) $\varkappa_{y_1}, \ldots, \varkappa_{y_n}$ is a K-basis of $\mathrm{End}_k K$ for dimensional reasons. Now, if $\varphi \in \mathrm{End}_{k-\mathrm{Alg}} K$, then $\varphi = \sum_{i=1}^n a_i \varkappa_{y_i}$ and hence using lemma on characters we get $\varphi = \varkappa_{a_i}\varkappa_{y_i} = \varkappa_{a_i y_i}$ for every index i with $a_i \neq 0$.)

3). Let $n := \mathrm{Dim}_k K$ be finite. Further, let $y_1, \ldots, y_n$ be an arbitrary k– basis of K and λ_i resp. ϱ_i be the left resp. right multiplications by y_i in K. Then $\lambda_i \varrho_j = \varrho_j \lambda_i$, $1 \leq i, j \leq n$, form a k–basis of $\mathrm{End}_k K$. (**Hint:** Let $f \in \mathrm{End}_k K = K\varkappa_{y_1^{-1}} + \cdots + K\varkappa_{y_n^{-1}}$ by part 1). Then for appropriate elements $\alpha_{ij} \in k$ we get $f = \sum_{i=1}^n a_i \varkappa_{y_i^{-1}} = \sum_{i,j=1}^n \alpha_{ij} y_j \varkappa_{y_i^{-1}} = \sum_{i,j=1}^n \alpha_{ij} \lambda_{y_j} \lambda_{y_i^{-1}} \rho_{y_i} \in \sum_{i,j=1}^n k \lambda_{y_j} \rho_{y_i}$.

Remark: The part 3) could be reformulated without any use of coordinates in the following way: The canonical k-algebra-homomorphism

$$K \otimes_k K^{\mathrm{op}} \longrightarrow \mathrm{End}_k K\ , \qquad x \otimes y \longmapsto \lambda_x \rho_y\ ,$$

is an isomorphism. This shows directly that K^{op} is the inverse of K in the Brauer group of k. – Moreover, $\mathrm{End}_{K'} K = \mathrm{C}(K') \otimes_k K^{\mathrm{op}}$ for all k-subalgebras $K' \subseteq K$, where $\mathrm{C}(K')$ is the algebra containing the elements of K commuting with all elements of K'.)

f). Every ring –endomorphism of the quaternion $\mathbb{H}(\mathbb{R})$ algebra is an inner automorphism. (**Hint:** Note that $\mathbb{R} = \mathrm{Z}(\mathbb{H}(\mathbb{R}))$ by exercise 5.13 and the only automorphism of $\mathbb{R}$ is identity. Therefore every ring endomorphism of $\mathbb{H}(\mathbb{R})$ is an $\mathbb{R}$-algebra endomorphism of $\mathbb{H}(\mathbb{R})$ and hence by the theorem of Noether–Skolem– Brauer, it is an inner automorphism.) The same result hold for $\mathbb{H}(\mathbb{Q})$.

* **7.10.** Let A be a ring, V be a free A–module with a basis x_i, $i \in I$, and $B := \mathrm{End}_A V$ be the A–endomorphism ring of V. Let $\mathfrak{U}$ denote the set of all those submodules of V which have generating system consisting of at most $|I|$ elements.

a). A submodule U of V belong to $\mathfrak{U}$ if and only if there exists $f \in B$ with $\mathrm{im}\, f = U$.

b). The map defined by $fB \mapsto \mathrm{im}\, f$ is a bijective map from the set of right-principal ideals in B onto the set $\mathfrak{U}$. (**Hint:** For $f, g \in B$, we have $\mathrm{im}\, f \subseteq \mathrm{im}\, g$ if and only if $f \in gB$.)

c). In the following cases $\mathfrak{U}$ is the set of all submodules of V:

1). I is infinite and $|I| \geq |A|$. (**Hint:** By the [4]) we have $|V| \leq |I|$.)

2). I is infinite and A is left-noetherian. (**Hint:** see the exercise [5]) which will be added in the exercise set on noetherian modules).

3). A is a left-principal ideal ring. (**Hint:** If I is finite, then apply the exercise [6]) which will be added in the exercise set on noetherian modules.)

d). Let $f_1, \ldots, f_r \in B$ with $U := \mathrm{im}\, f_1 + \cdots + \mathrm{im}\, f_r \in \mathfrak{U}$. Then there exists an element $h \in f_1 B + \cdots + f_r B$ such that $\mathrm{im}\, h = U$. Further, $hB = f_1 B + \cdots + f_r B$ for every $h \in B$ with $\mathrm{im}\, h = U$. (**Hint:** Let y_i, $i \in I$, be a system of generators of U and $y_i = \sum_{j=1}^r f_j(v_{ij})$ with $v_{ij} \in V$. Let h_j be defined by $x_i \mapsto f_j(v_{ij})$ and $h := h_1 + \cdots + h_r$.)

---

[4]) **Theorem** *Let A be a ring and let V be a free A–module with infinite rank. Then*

$$|V| = |A| \cdot \mathrm{rank}_A V = \mathrm{Sup}\{|A|,\ \mathrm{rank}_A V\}\,.$$

[5]) **Exercise** Let V be a module over a left-noetherian ring A and let $x_i, i \in I$ be a generating system for V, where I is infinite. Then every submodule U of V has a generating system of the form $y_i, i \in I$.

[6]) **Exercise** Let A be a ring in which every left- ideal has a generating system consisting of r elements and let V be an A–module generated by n elements. Then every submodule U of V is generated by nr elements. (**Hint:** By Induction on N. Suppose that $V = Ax_1 + \cdots + Ax_n$ and $f : V \to V/Ax_1$ be the residue map. Now consider $f|U$.) **Corollary:** *Over a left-principal ideal ring, every submodule of a module with n generators has a generating system consisting of n elements.*

e). If I is infinite, then every finitely generated right-ideal in B is a right- principal ideal. (**Hint:** Use the set-theoretic observation [7]) and the part d) of this exercise.)

f). Suppose that I is finite and $f_j,\ j \in J$, is a family of elements from B. Then the right-ideal $\sum_{j\in J} f_j B$ generated by $f_j,\ j \in J$ is the unit ideal if and only if $\sum_{j\in J} f_j(V) = V$. (**Hint:** Use d).

g). Suppose that $f_1, \ldots, f_r \in B$ with $W := \operatorname{im} f_1 \cap \cdots \cap \operatorname{im} f_r \in \mathfrak{U}$. Then $gB = f_1 B \cap \cdots \cap f_r B$ for every $g \in B$ with $\operatorname{im} g = W$.

* **7.11.** Let A be a left-principal ideal ring, V be a free A–module and let $B := \operatorname{End}_A V$ be the endomorphism ring of V. Then:

a). Sum and intersection of finitely many right-principal ideals in B is again a right-principal ideal in B. (**Hint:** Use 7.10, c)-3) and d), g).

b). The map $fB \mapsto \operatorname{im} f$ is an isomorphism of the lattice of the right-principal ideals of B onto the lattice of the A–submodules of V. (**Hint:** Choose a basis of V. Then $\mathfrak{U}$ is the set of all submodules of V by 7.10 c)-3). Now, apply the exercise 7.10.)

c). (Theorem of E. Noether) Suppose that V is a finite free A–module. Then the endomorphism ring $\operatorname{End}_A V$ is a right-principal ideal ring. (**Hint:** Choose a basis for V and use the exercise 7.10 with the following observation: *Let $\mathfrak{b}$ be a right-ideal in $B := \operatorname{End}_A V$. There exist $f_1, \ldots, f_r \in \mathfrak{b}$ such that $\sum_{f\in\mathfrak{b}} \operatorname{im} f = \operatorname{im} f_1 + \cdots + \operatorname{im} f_r$. Then $\mathfrak{b} = f_1 B + \cdots + f_r B$.* — **Remark.** *Let A be a right-principal ideal ring and let V be a finite free A–module. Then the endomorphism ring $\operatorname{End}_A V$ is a left-principal ideal ring.* **Proof** Choose a basis of V consisting of n elements. Then $\operatorname{End}_A V \cong \mathrm{M}_n(A^{\mathrm{op}}) = \mathrm{M}_n(A)^{\mathrm{op}}$ is left-principal ideal ring if and only if $\mathrm{M}_n(A)$ is right-principal ideal ring. But $\mathrm{M}_n(A) = \mathrm{M}_n((A^{\mathrm{op}})^{\mathrm{op}}) = \operatorname{End}_{A^{\mathrm{op}}}((A^{\mathrm{op}})^n)$ is the endomorphism ring of the free A^{op}–module $(A^{\mathrm{op}})^n$ over the left-principal ideal ring A^{op}, and so by the theorem of E. Noether (part c) above) is a right-principal ideal ring. For the case when A is a division ring, see also the next exercise 7.12.)

7.12. Let K be a division ring, V be a K–vector space and let B be the endomorphism ring $\operatorname{End}_K V$.

a). For every subspace $U \subseteq V$, there exists a $f \in B$ such that $\ker f = U$.

b). For $f, g \in B$, we have $\ker f \subseteq \ker g$ if and only if $g \in Bf$. Deduce that: for $f, g \in B$, we have $\ker f = \ker g$ if and only if $Bf = Bg$.

c). For $f_1, \ldots, f_r \in B$, $Bh = Bf_1 \cap \cdots \cap Bf_r$ for all $h \in B$ with $\ker h = \ker f_1 + \cdots + \ker f_r$.

d). For $f_1, \ldots, f_r \in B$, $Bg = Bf_1 + \cdots + Bf_r$ for all $g \in B$ with $\ker g = \ker f_1 \cap \cdots \cap \ker f_r$. (**Hint:** Using induction reduce to the case $r = 2$. Then use the exercise 4.15.)

e). The map $Bf \mapsto \ker f$ is an anti–isomorphism of the lattice of the left-principal ideals of B onto the lattice of the K–subspaces of V.

f). If V is finite dimensional, then $B = \operatorname{End}_K V$ is a left-principal ideal ring. (**Remark:** B is also a right-principal ideal ring by exercise 7.11 c).)

7.13. A group G is called rigid if identity map id_G is the only automorphism of G is the identity map of G. A group G is rigid if and only if $|G| \leq 2$. (**Hint:** First show that every rigid group is an elementary abelian 2-group. See T7.17)

7.14. Let V be a non-zero abelian group which is the additive group of a $\mathbb{R}$–vector space of finite dimension n. Neither the $\mathbb{R}$–vector space structure on V nor its dimension n are enough to determine V (as shown in the [8]) In addition to this there exists many different $\mathbb{R}$–vector space structures on V with the same dimension n. In the following we give some methods for this construction.

a). The case $n \geq 3$ is simple. There exists a natural number $m \in \mathbb{N}^*$ with $n - m \geq 1$ and $m \neq n - m$. Let h be an isomorphism (exists by the theorem below) between $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$.

[7]) *Let I be an infinite set and let $Y_j,\ j \in J$ be a family of sets with $|J| \leq |I|$ and $|Y_j| \leq |I|$ for all $j \in J$. Then $|\cup_{j\in J} Y_j| \leq |I|$.*

[8]) T7.17-2.c), 2d)

Now, in $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ with the help of h interchange the vector space structures of the direct summands.

b). Let $n = 2$. Then $\mathbb{R} = \mathbb{Q}^{(I)}$ with an indexed set I. Further, write $I = \{a\} \cup J$ with $a \notin J$, then

$$(\mathbb{Q}^{(J)} \oplus \mathbb{Q}^{\{a\}}) \oplus \mathbb{Q}^{(I)} = \mathbb{Q}^{(J)} \oplus (\mathbb{Q}^{\{a\}} \oplus \mathbb{Q}^{(I)}) ,$$

and both these summand representations essentially yield different decompositions of the form $\mathbb{R} \oplus \mathbb{R}$.

c). If $n = 1$ then one can choose a Hamel-basis of $\mathbb{R}$ which contain the number 1. Let α be a $\mathbb{Q}$–automorphism of $\mathbb{R}$ which interchanges 1 with another basis element in the Hamel basis and let $\varkappa_\alpha$ be the inner automorphism of $\operatorname{End}\mathbb{R}$ corresponding to α. If $\vartheta : \mathbb{R} \to \operatorname{End}\mathbb{R}$ is the canonical representation of $\mathbb{R}$, then one can give another $\mathbb{R}$–vector space structure on $\mathbb{R}$ via $\varkappa_\alpha\vartheta$. — With this method one can also spilt $\mathbb{R}$ into direct summands which will also cover the above cases.

* **7.15.** Let V be *not* finite dimensional vector space over a division ring K with $\operatorname{Dim}_K V = \alpha$ ($\geq \aleph_0$). Then the maps

$$\beta \mapsto \{f \in \operatorname{End}_K V : \operatorname{rank} f < \beta\} \text{ and } \mathfrak{b} \mapsto \operatorname{Min}\{\gamma : \operatorname{rank} f < \gamma \ \text{ for all } \ f \in \mathfrak{b}\}$$

are inverse isomorphisms to each other from the (well ordered) set of infinite cardinal numbers $\beta \leq \alpha$ and the set (ordered by the inclusion) of two-sided ideals $\mathfrak{b} \subseteq \operatorname{End}_K V$ with $0 \neq \mathfrak{b} \neq \operatorname{End}_K V$. — How many two-sided ideals are there in the ring $\operatorname{End}(\mathbb{R}, +) = \operatorname{End}_{\mathbb{Q}} \mathbb{R}$? (— Basic results on cardinal numbers are needed!.)

Below one can see (simple) test-exercises.

Test-Exercises

T7.1. Let V and W be modules over a non-commutative ring A. Then the abelian group $\operatorname{Hom}_A(V, W)$), need not be an A–submodules of W^V. (For example, $\operatorname{Hom}_A(A, A)$ is not an A–submodule of A^A ; choose $a, c \in A$ with $ca \neq ac$. Then $f := \operatorname{id}_A \in \operatorname{Hom}_A(A, A)$, but $cf \notin \operatorname{Hom}_A(A, A)$, since $(cf)(a \cdot 1_A) = ca$, but $a \cdot ((cf)(1_A)) = ac$.)

T7.2. Let V and W be isomorphic modules over a ring A. Then $\operatorname{End}_A V$ and $\operatorname{End}_A W$ are isomorphic rings. If A is commutative, then $\operatorname{End}_A V$ and $\operatorname{End}_A W$ are isomorphic A–algebras. (**Hint:** Consider $\operatorname{End}_A V \to \operatorname{End}_A W$ with $f \mapsto hfh^{-1}$, where $h : V \to W$ is an A–isomorphism.)

T7.3. Let V be a vector space over a division ring K, U be a subspace of V and let $x \in V$, $x \notin U$. Then there exists a K–linear form f on V such that $f(U) = 0$ and $f(x) \neq 0$.

T7.4. Let A be an integral domain and let V be an A–module. If V is a torsion module, then $\operatorname{Hom}_A(V, A) = 0$.

T7.5. Let A be an integral domain with quotient field K. Then $\operatorname{Hom}_A(K, A) \neq 0$ if and only if $A = K$. ($f \in \operatorname{Hom}_A(K, A)$ is the homothecy of K with $f(1)$.) If K is a finite A–module, then $A = K$. (see exercise 3.5. –Moreover, if K is a submodule of an arbitrary direct sum of finite A–modules, then $A = K$.)

T7.6. Let K be a divison ring, V be a K–vector space of dimension ≥ 2 and $\varphi : V \to V$ be a group homomorphism with $\varphi(Kx) \subseteq Kx$ for all $x \in V$. Then φ is a homothecy. (**Hint:** First consider $\varphi(x), \varphi(y), \varphi(x + y)$, where $x, y \in V$ are linearly independent.)

T7.7. (R e s t r i c t i o n o f m o d u l e s t r u c t u r e) Let A, B be rings, $\varphi : A \to B$ be a ring homomorphism and let W be a B–module. *With the help of φ, from the B–module structure on W, we can define an A–module structure on W.* Define the operation of A on W by $ay := \varphi(a)y$ for $a \in A$ and $y \in W$. If one thinks that the B–module structure on W is given by the ring homomorphism $\vartheta : B \to \operatorname{End} W$, then the above A–module structure on W is given by the composition $\vartheta\varphi : A \to \operatorname{End} W$. This A-module structure on W is said to be obtained the r e s t r i c t i o n o f s c a l a r s f r o m B t o A v i a φ or just the i n d u c e d m o d u l e s t r u c t u r e b y φ .

A simple example of the restriction of scalars of the B–module W is onto a subring B' of B. In this case φ is the canonical inclusion $B' \to B$. Another example is the canonical k–module structure of a module V over a k–algebra A. In this case φ is the structure homomorphism $k \to A$ of A.

Let $\varphi : A \to B$ be a surjective ring homomorphism. If V, W are modules over B, then $\operatorname{Hom}_B(V, W) = \operatorname{Hom}_A(V, W)$. (The B–modules are considered as A-modules by the restriction of scalarsvia φ.)

T7.8. Let A be a ring and let U, W be submodules of an A–module. Suppose that $f : U \to X$ and $g : W \to X$ are homomorphisms of A–modules into an A–module X. Then there exists a homomorphism $h : U + W \to X$ such that $h|U = f$ and $h|W = g$ if and only if $f|U \cap W = g|U \cap W$.

T7.9. Let A be a non-zero ring and let V be a free A–module which has a basis consiting of atleast two elements. Then the endomorphism ring $\mathrm{End}_A V$ is neither commutative nor free from zero-divisors.

T7.10. Let K be a field and let f resp. g be the K–endomorphisms of K^3 defined by $e_1 \mapsto e_2$, $e_2 \mapsto e_1$, $e_3 \mapsto 0$ resp. $e_1 \mapsto 0$, $e_2 \mapsto e_3$, $e_3 \mapsto e_2$, where e_1, e_2, e_3 is the standard basis of K^3. Then : (1) $\mathrm{im}\, f + \mathrm{im}\, g = K^3$. (2) $\ker f \cap \ker g = 0$. (3) No linear combination of f and g is an automorphism of K^3.

T7.11. Let V be a finite dimensional vector space over a division ring K and let $f \in \mathrm{End}_K V$. **a).** The following statements are equivalent: (1) $\mathrm{im}\, f = \ker f$. (2) $f^2 = 0$ and $\mathrm{Dim}_K V = 2 \cdot \mathrm{rank}\, f$.

b). The following statements are equivalent: (1) $\mathrm{rank}\, f = \mathrm{rank}\, f^2$. (2) $\mathrm{im}\, f = \mathrm{im}\, f^2$. (3) $\ker f = \ker f^2$. (4) $\mathrm{im}\, f \cap \ker f = 0$. (5) $\mathrm{im}\, f + \ker f = V$.

T7.12. Let A be a finite dimensional algebra over a field K. For an element $x \in A$ the following statements are equivalent: (1) x is a unit in A. (2) x is not a left -zero divisor in A. (2′) x is not a right-zero divisor in A. (3) x has a left-inverse. (3′) x has a right-inverse. (**Hint :** Consider the translations λ_x and ϱ_x in A with x.) — Use this to give a new proof of exercise 5.17.)

T7.13. Let A be a finite dimensional algebra over a field K. If $x \in A$ is unit, then x^{-1} belong to the K–subalgebra $K[x]$ of A generated by x.

T7.14. (Bimodules) Often we consider many module structures on the same abelian group V at the same time. If these module structures are *compatible* with each other, then we can use the term Multi-modules, especially Bimodules, if two compatible modules structures on the same abelian group are considered.

Let V be an abelian group. Suppose that V has an A– as well as B–(left–)module structure. Then V is called an (A, B)–bimodule if $a(bx) = b(ax)$ for all $a \in A$, $b \in B$, $x \in V$. This is usually writen as $V = {}_{A,B}V$.

Suppose that V has an A–left – and a B–right module structure. Then V is called an (A, B)–bimodule, if $a(xb) = (ax)b$ for all $a \in A$, $b \in B$, $x \in V$. In this case we use the notation $V = {}_AV_B$.

Analogously, bimodules of the type $V_{A,B}$ can be defined.

a). A trivial example of a bimodule structure is an usual module V over a *commutative* ring A. With one and the same operation V is an (A, A)–bimodule of the type ${}_{A,A}V$.

b). Let V be an A–module. Then V has a natural $(\mathrm{End}_A V)$–module structure given by the operation $(f, x) \mapsto f(x)$, for $f \in \mathrm{End}_A V$ and $x \in V$. (Note that Giving an A–module structure on the abelian group V is equivalent to giving a (canonical) ring homomorphism $\vartheta : A \to \mathrm{End}_{\mathbb{Z}} V$. Further, the ring $B := \mathrm{End}_A V$ is a subring of the ring $\mathrm{End}_{\mathbb{Z}} V$ and hence the $(\mathrm{End}_A V)$–module structure on V is nothing but the restriction of scalars from the ring $\mathrm{End}_{\mathbb{Z}} V$ to the subring $\mathrm{End}_A V$.) Therefore V is an A–module as well as B–module and further V is a (A, B)–module of type ${}_{A,B}V$; since for $a \in A$, $f \in B$ and $x \in V$, we have $a(f \cdot x) = af(x) = f(ax) = f \cdot (ax)$.

T7.15. (Homomorphism modules of bimodules) We have seen in T7.1 that in general, the abelian group $\mathrm{Hom}_A(V, W))$ need not be an A–submodules of W^V. Let B be an another ring. We are looking for some conditions either on V or on W, so that the abelian group $\mathrm{Hom}_A(V, W))$ has a B–module structure. For example :

a). *Let V be an A–module and W be a (A, B)–bimodule of type ${}_AW_B$. Then the abelian group $\mathrm{Hom}_A(V, W)$ is a B–right module with respect to the right operation of B on $\mathrm{Hom}_A(V, W)$ defined by*

$$fb := (x \mapsto f(x)b) \ , \ b \in B \text{ and } f \in \mathrm{Hom}_A(V, W) .$$

(**Proof** We consider W^V as B–right module, where for $b \in B$ and $f \in W^V$ the product fb is defined by $fb := (x \mapsto f(x)b)$. We need therefore to show that the subset $\mathrm{Hom}_A(V, W)$ of W^V is a B–right submodule. Since $\mathrm{Hom}_A(V, W)$ is a subgroup of W^V, it is enought to show that $\mathrm{Hom}_A(V, W)$ is closed (in W^V) under the right-operation of B and this can be easily verified. •

b). *If V is an A–module and W be a (A, B)–bimodule of type ${}_{A,B}W$, then the abelian group $\mathrm{Hom}_A(V, W)$ is a B–(left) module with respect to the (left) operation of B on $\mathrm{Hom}_A(V, W)$ defind by*

$$bf := (x \mapsto bf(x)) \ , \ b \in B \text{ and } f \in \mathrm{Hom}_A(V, W) .$$

An important example for the application of a) is: If V is an A–module, then the abelian group $V^* = \mathrm{Hom}_A(V, A)$ has a natural A-right module structure, since A is trivially a (A, A)–bimodule of type ${}_AA_A$. With this right module structure the module V^* is called the d u a l m o d u l e of V. If A is commutative, then this module structure is same as that of the natural A–modules strukture, namely $af := (x \mapsto af(x))$ for $a \in A$ and $f \in \mathrm{Hom}_A(V, A)$.

c). *Let V be a (A, B)–bimodule of type ${}_AV_B$ and let W be an A–module. Then the abelian group* $\mathrm{Hom}_A(V, W)$ *is a B–(left) module with respect to the (left) operation of B on* $\mathrm{Hom}_A(V, W)$ *defined by*

$$bf := (x \mapsto f(xb)) \ , \ b \in B \text{ and } f \in \mathrm{Hom}_A(V, W) .$$

(**Proof** The multiplicative monoid of B operates on W^V in a canonical way: $bf = (x \mapsto f(xb))$ for $b \in B$ and $f \in W^V$. We need to show that this operation cane be restricated to $\mathrm{Hom}_A(V, W)$. Thereofre, let $b \in B$, $f \in \mathrm{Hom}_A(V, W)$ and $x, y \in V$. Then $(bf)(x+y) = f((x+y)b) = f(xb+yb) = f(xb)+f(yb) = (bf)(x)+(bf)(y)$. Further, for $a \in A$, we have $(bf)(ax) = f((ax)b) = f(a(xb)) = af(xb) = a((bf)(x))$. Therefore $bf \in \mathrm{Hom}_A(V, W)$, and so the given operation of B on $\mathrm{Hom}_A(V, W)$ exists. Now, one can easily verify that this operation defines a B–module structure on $\mathrm{Hom}_A(V, W)$. •

d). *If V is a (A, B)–bimodule of type ${}_{A,B}V$ and W is an A–module, then the abelian group* $\mathrm{Hom}_A(V, W)$ *is a B–right module with respect to the right operation of B on* $\mathrm{Hom}_A(V, W)$ *defined by*

$$fb := (x \mapsto f(bx)) \ , \ b \in B \text{ and } f \in \mathrm{Hom}_A(V, W) .$$

T7.16. Let A be a ring, V be an A–module with basis x_i, $i \in I$ and let W be an arbitrary A–module. Then the map

$$\sigma : \mathrm{Hom}_A(V, W) \to W^I$$

defined by $f \mapsto (f(x_i))_{i \in I}$ is bijective. Further,

a). σ is an isomorphism of the additive groups. (**Proof** For $f, g \in \mathrm{Hom}_A(V, W)$ $\sigma(f+g) = ((f+g)(x_i)) = (f(x_i) + g(x_i)) = (f(x_i)) + (g(x_i)) = \sigma(f) + \sigma(g)$.)

b). Let W be an (A, B)–bimodule with respect to another ring B. *Then σ is an isomorphism of B–modules.* (**Proof** Suppose for example W is an (A, B)–bimodule of type ${}_AW_B$. Then for $b \in B$ and $f \in \mathrm{Hom}_A(V, W)$ we have $\sigma(fb) = ((fb)(x_i)) = (f(x_i)b) = (f(x_i))b = \sigma(f)b$. Therefore σ is an isomorphism of B–right-modules.) Similarly, σ is an isomorphism of B–(left)-modules, if W is an (A, B)–bimodule of type ${}_{A,B}W$. Every A–module is trivially $(A, \mathrm{Z}(A))$–bimodule. Therefore σ is linear over the center $\mathrm{Z}(A)$ of A. In particular: *If A is commutative, then σ is an isomorphism of A–modules.*

c). An important special case is $W = A$. Since A is an (A, A)–bimodule of type ${}_AA_A$, we have : *Then canonical map of* $V^* = \mathrm{Hom}_A(V, A)$ *in A^I is an isomorphism of A–right-modules*. In particular, in the case $V = A^{(I)}$ we have a canonical isomorphism $(A^{(I)})^* \to A^I$ of A–right-modules. In the general case if $V = A^{(I)}$. The direct sum $A^{(I)}$ with the standard basis e_i, $i \in I$, is a (A, A)–bimodule of type ${}_A(A^{(I)})_A$ in a canonical way. Therefore $\mathrm{Hom}_A(A^{(I)}, W)$ has a natural A–module structure by T7.15-c). *with respect to this structure and the canonical module structure on W^I, σ is always an isomorphism*. For, if $a \in A$ and $f \in \mathrm{Hom}_A(A^{(I)}, W)$, then $\sigma(af) = ((af)(e_i)) = (f(e_i a)) = (f(ae_i)) = (af(e_i)) = a(f(e_i)) = a\sigma(f)$. In particular, the canonical isomorphism $\mathrm{Hom}_A(A, W) \to W$ of A–modules is given by $f \mapsto f(1)$.

d). In the case $W = A$ we have : The *canonical map $(A^{(I)})^* \to A^I$ is an isomorphism of (A, A)–bimodules*. In words: dualising converts the direct sums $A^{(I)}$ into the direct product A^I in a canonical way.

T7.17. (A b e l i a n g r o u p s w i t h v e c t o r s p a c e s t r u c t u r e) We would like to describe the abelian groups which are the underlying additive groups of vector spaces over division rings. Each vector space over a division ring K, by restricting the scalars, we consider a vector space over the prime field of K. Since prime fields are canonically isomorphic to $\mathrm{K}_p = \mathbb{Z}/p\mathbb{Z}$ in the case of prime characteristic p resp. to $\mathbb{Q}$ in the case of characteristic 0. With this we need to describe those abelian groups which are underlying additive groups of the vector spaces over the fields K_p resp. $\mathbb{Q}$. For this we shall use different methods for different characteristic.

1). First consider the case of a prime characteristic. Let p be a prime number. An additively written abelian group H is called an *elementary abelian p–group*, if $px = 0$ for all $x \in H$, therefore every element of H is of order 1 or p. (An elementary abelian 2–group is nothing but the group in which every element is its self inverse.)

1.a) *An abelian group is the additive group of a K_p–vector space if and only if it is an elementary abelian p–group. Moreover, in this case the K_p–vector space–structure is uniquely determined.* (**Proof** If V is a vector space over K_p, then $px = 0$ for all $x \in V$. Conversely, suppose that H is an elementary abelian

p–group. By assumption on H the kernel of the canonical ring homomorphism $\chi : \mathbb{Z} \to \mathrm{End}_{\mathbb{Z}} H$ defined by $n \mapsto n \cdot \mathrm{id}_H$ contains the prime number p and hence contains $p\mathbb{Z}$. Therefore χ induces a homomorphism χ_p of $\mathrm{K}_p = \mathbb{Z}/p\mathbb{Z}$ in $\mathrm{End}_{\mathbb{Z}} H$, which defines a K_p–vector space–structure on H. Since K_p is a prime ring, this is the only homomorphism of the ring K_p into $\mathrm{End}_{\mathbb{Z}} H$ and hence the given vector spaces structure on H is uniquely determined.)

Let H be an elementary abelian p–group. A subgroup H' of H is always an elementary abelian p–group and hence K_p–vector space in a canonical way. This vector space structure is clearly the restriction of the vector space structure from H; therefore H' is a subspace of H. From now on every elementary abelian p–group is considered as K_p–vector space.

1.b) *For elementary abelian p–group H, F we have* $\mathrm{Hom}(H, F) = \mathrm{Hom}_{\mathrm{K}_p}(H, F)$. **Proof** We have $\mathrm{Hom}(H, F) = \mathrm{Hom}_{\mathbb{Z}}(H, F)$. Now, apply T7.7 in the situation $\mathbb{Z} \to \mathrm{K}_p$.)

With 1.a) and 1.b) the theory of the elemenatray abelian p–groups is equivalent to the theory of the K_p–vector spaces. As an applications can be directly given. For example every elementary abelian p–group H is isomorphis to a group of the form $\mathrm{K}_p^{(I)}$, where the cardinality of I is uniquely determined, namely as the dimension of the K_p–vector space H, vgl. Beispiel 1. Alternatively, $|I|$ can be simply computed from $|H|$. For example if H is finite, then $|I|$ is determined by $|H| = p^{|I|}$. If H is infinite, then $|I| = |H|$ by footnote 4. In particular, we have: *An elementary abelian p–group is uniquely determined upto isomorphism by its cardinality*. For further applications see ???).

2). We now consider the abelian groups which has a $\mathbb{Q}$–vector space structure. An (additively written) abelian group H is called t o r s i o n f r e e, if the homothecies $x \mapsto nx$ for $n \in \mathbb{Z}$, $n \neq 0$, are injective, and is called d i v i s i b l e, if the homothecies are surjective.

2.a) *An abelain group is the additive group of a $\mathbb{Q}$–vector space if and only if it is torsion free and is divisible. Moreover, in this case the $\mathbb{Q}$–vector space–structure is uniquely determined.* (**Proof** If V is a $\mathbb{Q}$–vector space and $n \in \mathbb{Z}$, $n \neq 0$, then the homothecy ϑ_n by n is bijective on V. But ϑ_n is the multiplication $x \mapsto nx$ by n on the $\mathbb{Z}$–module V. Therefore it follows that V is torsion free and divisible. Conversely, suppose that H is torsion free divisible abelian group. We may assume that $H \neq 0$. The homothecies $x \mapsto nx$, $n \neq 0$, in $\mathrm{End}_{\mathbb{Z}} H = \mathrm{End}\ H$ are by hypothesis are automorphisms of H. Therefore are the multiples $n \cdot \mathrm{id}_H$ of the unit element of $\mathrm{End}\ H$ and hence units in this endomorphism ring. Since $H \neq 0$, $\mathrm{End}\ H$ contains a field of characteristic 0 and there exists a unique homomorphism $\chi : \mathbb{Q} \to \mathrm{End}\ H$. With this there exists a unique $\mathbb{Q}$–vector space structure on H.)

From now on every torsion free divisible abelian group is considered as a $\mathbb{Q}$–vector space.

2.b) *For torsion free divisible abelian groups H, F we have* $\mathrm{Hom}(H, F) = \mathrm{Hom}_{\mathbb{Q}}(H, F)$. (**Proof** Let $f \in \mathrm{Hom}(H, F) = \mathrm{Hom}_{\mathbb{Z}}(H, F)$. We need to show that f is linear over $\mathbb{Q}$. For this let $x \in H$ and $q = m/n \in \mathbb{Q}$, $m, n \in \mathbb{Z}$, $n \neq 0$, then $nf(qx) = f(n(qx)) = f((nq)x) = f(mx) = mf(x) = (nq)f(x) = n(qf(x))$. Cancelling n we get $f(qx) = qf(x)$, as desired.)

With 2.a) and 2.b) the theory of torsion free divisible abelain groups is eqivalent to the theory of the $\mathbb{Q}$–vector spaces. As an application we can give a simple classification of the torsion free divisible abelain groups. Every such group H is isomorphic to $\mathbb{Q}^{(I)}$, where I is a set with $|I| = \mathrm{Dim}_{\mathbb{Q}} H$. This cardinal number uniquely determines the group upto isomorphism. If H is countable, then I is necessarily finite or countably infinite. Therefore we have: *The isomorphism type of the countable torsion free divisible abelain groups are represented by the groups $\mathbb{Q}^n$, $n \in \mathbb{N}$, and $\mathbb{Q}^{(\mathbb{N})}$.* If H is uncountable, then I is infinite and hence $|H| = |\mathbb{Q}^{(I)}| = |I|$ by footnote 4. Therefore we have: *An uncountable torsion free divisible abelian group is uniquelly determined upto isomorphism by its cardinality.*

We further note the following applications.

2.c) *Let K be a division ring which is not finite dimensional over its prime field. Then the additive groups of all finite non-zero K–vector spaces are isomorphic to each other.* (**Proof** Let k be the prime field of K. For a n–dimensional K–vector space V, $n \geq 1$, we have (see footnote 4) $\mathrm{Dim}_k V = n \cdot \mathrm{Dim}_k K = \mathrm{Dim}_k K$ and this cardinal number does not depend on n.)

2.d) *The additive groups $\mathbb{R}^n$, $\mathbb{C}^n$, $n \in \mathbb{N}^*$, are isomorphic to each other.* (**Proof** $\mathbb{C}^n$ is a $\mathbb{R}$–vector space of dimension $2n$ and $\mathbb{R}$ is an infinite dimensional over $\mathbb{Q}$. Now apply the above theorem.)

2.e) *The endomorphism ring* $\mathrm{End}\ H$ *of an abelain group H is a division ing if and only if H is the additive group of a prime field, i.e. if H is cyclic of prime order or if H isomorphic to* $(\mathbb{Q}, +)$. (**Proof** Suppose that $\mathrm{End}\ H$ is a division ring. If k is a prime field of $\mathrm{End}\ H$, then H is a k–module and by 1. b) resp. 2.b) $\mathrm{End}\ H = \mathrm{End}_k H$. By exercise T7.9, the last ring is a division ring if and only if $\mathrm{Dim}_k H = 1$.

Therefore H is isomorphic to the additive group of k. Conversely, suppose that H is cyclic of prime order p, then End $H = \mathrm{End}_{K_p} H$ by 1.a); if $H = (\mathbb{Q}, +)$, then End $H = \mathrm{End}_{\mathbb{Q}} H$ by 2.b). In both the cases the endomorphism rings are isomorphic to prime fields.)

---

† **Richard Dagobert Brauer (1901-1977)** was born on 10 Feb 1901 in Berlin-Charlottenburg, Germany and died on 17 April 1977 in Belmont, Massachusetts, USA. Richard Brauer's father was Max Brauer who was a well-off businessman in the wholesale leather trade. Max Brauer's wife was Lilly Caroline and Richard was the youngest of their three children. He had an older brother Alfred Brauer, who also became a famous mathematician. Alfred Brauer was seven years older than Richard and of an age between the two brothers was Richard's sister Alice.

Richard entered the Kaiser-Freidrich-Schule in Charlottenburg in 1907. Charlottenburg was a district of Berlin which was not incorporated into the city until 1920. Richard studied at this school until 1918 and it was during his school years that he developed his love of mathematics and science. However, this was not due to the teaching at the school, but it came about through the influence of his brother Alfred. Richard writes about his school teachers who he describes as not being very competent. There was one exception however, and it was fortunate that this good teacher was a mathematician who had a doctorate awarded for research done under Frobenius's supervision.

Of course Richard's last four years at school were the years of World War I, but, unlike his brother, he was young enough to avoid being drafted into the army. When he graduated from the Kaiser-Freidrich-Schule in September 1918 the war was still in progress, and Brauer was drafted to undertake civilian war service in Berlin. Only two months later, in November 1918, the war ended, Brauer was released from war service and he resumed his education. Despite the love for mathematics which he had gained from his brother, Brauer decided to follow his boyhood dreams of becoming an inventor. He entered the Technische Hochschule of Charlottenburg in February 1919 where he studied for a term before, having realised that his talents were in theory rather than practice, he transferred to the University of Berlin.

At the University of Berlin Brauer was taught by a number of really outstanding mathematicians including Bieberbach, Carathéodory, Einstein, Knopp, von Mises, Planck, Schmidt, Schur and Szego. Brauer describes some of the lectures he attended; talking of Schmidt's lectures he writes: *It is not easy to describe their fascination. When Schmidt stood in front of a blackboard, he never used notes, and was hardly ever well prepared. He gave the impression of developing the theory right there and then.*

It was the custom that German students at this time spent periods in several different universities during their degree course. Brauer was no exception to this, although he made only one visit during his studies, that being for a term to the University of Freiburg. Back in Berlin he attended seminars by Bieberbach, Schmidt and Schur. He was increasingly attracted towards the algebra which Schur was presenting in his seminar (which was attended in the same year by Alfred Brauer). Schur, unlike Schmidt,

... was very well prepared for his classes, and he lectured very fast. If one did not pay the utmost attention to his words, one was quickly lost. There was hardly any time to take notes in class; one had to write them up at home. ... He conducted weekly problem hours, and almost every time he proposed a difficult problem. Some of the problems had already been used by his teacher Frobenius, and others originated with Schur. Occasionally he mentioned a problem he could not solve himself.

In fact it was one of these open problems which Richard working with his brother Alfred solved in 1921 and this was eventually to be included in Brauer's first publication. Schur suggested the problem that Brauer worked on for his doctorate and the degree was awarded (with distinction) in March 1926. His dissertation took an algebraic approach to calculating the characters of the irreducible representations of the real orthogonal group. Before the award of his doctorate, however, Brauer had married Ilse Karger in September 1925. They had been a fellow students in one of Schur's courses on number theory. Before his marriage Brauer was appointed as Knopp's assistant at the University of Königsberg and he took up this post in the autumn of 1925.

Shortly after Brauer arrived in Königsberg, Knopp left to take up an appointment at Tübingen. The mathematics department at Königsberg was small, with two professors Szego and Reidemeister, and with Rogosinski and Kaluza holding junior positions like Brauer. It was in Königsberg that Brauer's two sons, George Ulrich Brauer and Fred Günter Brauer were born. Brauer taught at Königsberg until 1933 and during this period he produced results of fundamental importance. Green writes :

This was the time when Brauer made his fundamental contribution to the algebraic theory of simple algebras. ... Brauer developed ... a theory of central division algebras over a given perfect field, and showed that the isomorphism classes of these algebras can be used to form a commutative group whose properties gave great insight into the structure of simple algebras. This group became known (to the author's embarrassment) as the "Brauer group" ...

Political events forced Brauer's family to move. He wrote : *I lost my position in Königsberg in the spring of 1933 after Hitler became Reichskanzler of Germany.*

Brauer was from a Jewish family so was dismissed from his post under the Nazi legislation which removed all Jewish university teachers from their posts. This was a desperate time for Brauer who realised that he had to find a post outside Germany. Fortunately action was taken in several countries to find posts abroad for German academics forced from their positions and a one year appointment was arranged for Brauer in Lexington, Kentucky for the academic year 1933-34. In November 1933 Brauer arrived to take up his appointment at the University of Kentucky, his wife and two sons following three months later. We should record that Alfred Brauer left Germany in 1939, but Brauer's sister Alice stayed behind and was murdered in a concentration camp by the Nazis.

Following his year in Lexington, Brauer was appointed as Weyl's assistant at the Institute for Advanced Study in Princeton. He wrote about this appointment with Weyl: *I had hoped since the days of my PhD thesis to get in contact with him some day; this dream was now fulfilled.*

Collaboration between Brauer and Weyl on several projects followed, in particular a famous joint paper on spinors published in 1935 in the American Journal of Mathematics. This work was to provide a background for the work of Paul Dirac in his exposition of the theory of the spinning electron within the framework of quantum mechanics.

A permanent post followed the two temporary posts when Brauer accepted an assistant professorship at the University of Toronto in Canada in the autumn of 1935. It was largely as a result of Emmy Noether's recommendation, which she made while visiting Toronto, which led to his appointment. This was a time when Brauer developed some of his most impressive theories, carrying the work of Frobenius into a whole new setting, in particular the work on group characters Frobenius published in 1896. Brauer carried Frobenius's theory of ordinary characters (where the characteristic of the field does not divide the order of the group) to the case of modular characters (where the characteristic does divide the group order). He also studied applications to number theory.

C J Nesbitt was Brauer's first doctoral student in Toronto and he described their relationship as doctoral student and supervisor : *Curiously, as thesis advisor, he did not suggest much preparatory reading or literature search. Instead we spent many hours exploring examples of the representation theory ideas that were evolving in his mind.*

It was in joint work with Nesbitt, published in 1937, that Brauer introduced the theory of blocks. This he used to obtain results on finite groups, particularly finite simple groups, and the theory of blocks would play a big part in much of Brauer's later work.

Alperin also spoke of Brauer's thirteen years in Toronto : *The years he spent at Toronto were his most productive years. He achieved five or six great results during that time, any one of which would have established a person as a first-rank mathematician for the rest of their life. ... those years had their high points, but also contained fallow periods, when there was the day-to-day grind of raising a family in modest circumstances.*

Brauer spent 1941 at the University of Wisconsin having been awarded a Guggenheim Memorial Fellowship. He was the Colloquium lecturer at the American Mathematical Society Summer Meeting in Madison, Wisconsin in 1948. Later that year he moved from Toronto back to the United States, accepting a post at the University of Michigan in Ann Arbor. In 1949 Brauer was awarded the Cole Prize from the American Mathematical Society for his paper On Artin's L-series with general group characters which he published in the Annals of Mathematics in 1947. In 1951 Harvard University offered him a chair and, in 1952, he took up the position in Harvard which he was to hold until he retired in 1971. In the year of his retirement he was honoured with the award of the National Medal for Scientific Merit.

We have mentioned a number of topics which Brauer worked on in the course of this biography. However we have not yet mentioned the work which in many ways was his most famous and this he began around the time he took up the chair at Harvard. He began to formulate a method to classify all finite simple groups and his first step on this road was a group-theoretical characterisation of the simple groups $\mathrm{PSL}(2,q)$ in 1951 (although for a complicated number of reasons, this did not appear in print until 1958). This work was done jointly with his doctoral student K A Fowler, and in 1955 they published a major paper which was to set mathematicians on the road to the classification. The paper was On groups of even order and it provided the key to the major breakthrough by Walter Feit and John Thompson when they proved that every finite simple group has even order.

Brauer was to spend the rest of his life working on the problem of classifying the finite simple groups. He died before the classification was complete but his work provided the framework of the classification which was completed only a few years later. (See the biography of Gorenstein for further details on the programme to classify finite simple groups.) Most important was Brauer's vital step in setting the direction for the whole classification programme in the paper On groups of even order where it is shown that there are only finitely many finite simple groups containing an involution whose centraliser is a given finite group. Brauer had announced these results and his programme for classifying finite simple groups at the International Congress of Mathematicians in Amsterdam in 1954.

Green points out that when Brauer went to Harvard he was 51 years old, yet almost half his total of 147 publications were published after this date. He certainly did not sit quietly working away in Harvard. He spent extended periods visiting friends and colleagues in universities around the world, for example Frankfurt and Göttingen in Germany, Nagoya in Japan, and Newcastle and Warwick in England.

Despite his remarkable contributions to research, Brauer found time to act as an editor for a number of journals. He was an editor of the Transactions of the Canadian Mathematical Congress from 1943 to 1949, the American Journal of Mathematics from 1944 to 1950, the Canadian Journal of Mathematics from 1949 to 1959, the Duke Mathematical Journal from 1951 to 1956 and again from 1963 to 1969, the Annals of Mathematics from 1953 to 1960, the Proceedings of the Canadian Mathematical Congress from 1954 to 1957, and the Journal of Algebra from 1964 to 1970. A quick glance will show that in 1955 he held editorships of four learned journals.

We have mentioned above a number of honours which Brauer received. We should also mention the learned societies which honoured him with membership: the Royal Society of Canada (1945), the American Academy of Arts and Sciences (1954), the National Academy of Sciences (1955), the London Mathematical Society (1963), the Akademie der Wissenschaften Göttingen (1964), and the American Philosophical Society (1974). He was also elected President of the Canadian Mathematical Congress (1957-58) and the American Mathematical Society (1959-60).

Green describes Brauer's character (no pun intended!):

All who knew him best were impressed by his capacity for wise and independent judgement, his stable temperament and his patience and determination in overcoming obstacles. He was the most unpretentious and modest of men, and remarkably free of self-importance. ...

Brauer's interest in people was natural and unforced, and he treated students and colleagues alike with the same warm friendliness. In mathematical conversations, which he enjoyed, he was usually the listener. If his advice was sought, he took this as a serious responsibility, and would work hard to reach a wise and objective decision.

Richard Brauer occupied an honoured position in the mathematical community, in which the respect due to a great mathematician was only one part. He was honoured as much by those who knew him for his deep humanity, understanding and humility; these were the attributes of a great man.

---

† **Albert Thoralf Skolem (1887-1963)** was born 23 May 1887 in Sandsvaer, Norway and died on 23 March 1963 in Oslo, Norway. Thoralf Skolem worked on Diophantine equations, mathematical logic, group theory, lattice theory and set theory. In 1912 he produced a description of a free distributive lattice. He made refinements to Zermelo's axiomatic set theory, publishing work in 1922 and 1929.

Skolem extended work by Löwenheim (1915) to give the Löwenheim- Skolem theorem, which states that if a theory has a model then it has a countable model. From 1933 he did pioneering work in metalogic and constructed a nonstandard model of arithmetic. He also developed the theory of recursive functions as a means of avoiding the so-called paradoxes of the infinite.