D_M - 07 **MA-217 Discrete Mathematics / Jan-Apr 2007**
Lectures : Tuesday/Thursday 15:45–17:15 ; Lecture Hall-1, Department of Mathematics

1. Inclusion-Exclusion Principle

1.1. (Sylvester's Sieve--formula) Let $X_1, \ldots, X_n$ be finite sets. For $J \subseteq \{1, \ldots, n\}$, let $X_J := \bigcap_{i \in J} X_i$ with $X_\emptyset := \bigcup_{i=1}^n X_i$. Prove that

$$\sum_{J \in \mathfrak{P}(\{1,\ldots,n\})} (-1)^{|J|} |X_J| = 0\,, \quad \text{i.e.} \quad |X| = \sum_{\emptyset \neq J \in \mathfrak{P}(\{1,\ldots,n\})} (-1)^{|J|-1} |X_J|\,.$$

(**Hint :** By induction on n. — Variant: For $k = 1, \ldots, n$, let Y_k be the set of elements $x \in X_\emptyset$ which belong to exactly k of the sets $X_1, \ldots, X_n$. Then Y_k, $1 \leq k \leq n$ are pairwise disjoint. Using Exercise T1.3 b) show that

$$\sum_{\substack{J \in \mathfrak{P}(\{1,\ldots,n\}) \\ |J|\,\text{even}}} |X_J| = \sum_{k=1}^{n} 2^{k-1} |Y_k| = \sum_{\substack{J \in \mathfrak{P}(\{1,\ldots,n\}) \\ |J|\,\text{odd}}} |X_J|\,.)$$

1.2. a). Let X be a finite set with m elements. Let p_m denote the number of permutations of X which donot have fixed points and let $s_m = m!$ be the number of all permutations of X. Show that :

$$\frac{p_m}{s_m} = \frac{1}{0!} - \frac{1}{1!} + \cdots + (-1)^m \cdot \frac{1}{m!}\,.$$

(**Hint :** Let $X = \{x_1, \ldots, x_m\}$. Set $X_i := \{\sigma \in \mathfrak{S}(X) : \sigma(x_i) = x_i\}$ and compute $s_m - p_m = |\bigcup_{i=1}^m X_i|$ using the Sieve formula in Exercise 1.1. — **Remark :** Note that $\lim_{m\to\infty} (p_m/s_m) = e^{-1}$, where $e := \lim_{n\to\infty}(1+\frac{1}{n})^n = 2.71828182845904523536\ldots$ is the *Euler's number* which is base of the natural logarithm.) — The number of permutations of X with exactly r fixed points is $\binom{m}{r} p_{m-r}$, $0 \leq r \leq m$. (Proof!)

b). Let X be a finite set with m elements and let Y be a finite set with n elements. The number of surjective maps from X in Y is

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^n \binom{n}{n}(n-n)^m\,.$$

(**Hint :** Let $Y = \{y_1, \ldots, y_n\}$. Set $P_i := \{f \in Y^X : y_i \notin \operatorname{im} f\}$ and compute the number $|\bigcup_{i=1}^n P_i|$ of non-surjective maps using the Sieve formula in Exercise 1.1.)

1.3. Let I be a finite index set with n elements and let $\sigma_i \in \mathbb{N}$ for $i \in I$, $\pi := \prod_{i\in I} \sigma_i$, $\sigma := \sum_{i \in I} \sigma_i$ and $\sigma_H := \sum_{i \in H} \sigma_i$ for $H \subseteq I$. Then

$$\sum_{H \subseteq I} (-1)^{|H|} \binom{\sigma_H}{n} = (-1)^n \pi \quad \text{and} \quad \sum_{H \subseteq I} (-1)^{|H|} \binom{\sigma_H}{n+1} = \frac{(-1)^n}{2}(\sigma - n)\pi\,,$$

(**Hint :** Let $X = \bigcup_{i \in I} X_i$, where X_i are pairwise disjoint subsets with $|X_i| = \sigma_i$. For a proof of the first formula consider the set $\mathfrak{P}_n(X)$ and its subsets $Y_i := \{A \in \mathfrak{P}_n(X) \mid A \cap X_i = \emptyset\}$ and use the Sieve formula in Exercise 1.1 to find $|\bigcup_{i \in I} Y_i|$.

1.4. Let m, n be two natural numbers.

a). Let $\mathrm{a}(m,n)$ (resp. $\mathrm{b}(m,n)$) denote the number of m–tuples $(x_1, \ldots, x_m) \in \mathbb{N}^m$ with $x_1 + \cdots + x_m \leq n$ (resp. $x_1 + \cdots + x_m = n$). Show that $\mathrm{a}(m,n) = \binom{n+m}{m}$ and $\mathrm{b}(m,n) = \binom{n+m-1}{m-1}$. (**Hint :** Remember to put $\binom{-1}{-1} := 1$. Note that $\mathrm{a}(m-1,n) = \mathrm{b}(m,n)$ and $\mathrm{a}(m,n) = \mathrm{a}(m,n-1) + \mathrm{a}(m-1,n)$ if $m \geq 1$ and use induction on $n+m$. — Variant: The map $(x_1, \ldots, x_m) \mapsto \{x_1 + 1,\ x_1 + x_2 + 2, \ldots, x_1 + \cdots + x_m + m\}$ maps the set of m–tuples $(x_1, \ldots, x_m) \in \mathbb{N}^m$ with $x_1 + \cdots + x_m \leq n$ bijectively onto the set of m–element subsets of $\{1, 2, \ldots, n+m\}$.

b). Suppose that $m \geq 1$. Prove that the number of m–tuples $(x_1, \ldots, x_m) \in (\mathbb{N}^+)^m$ of positive natural numbers with $x_1 + \cdots + x_m = n$ is $\binom{n-1}{m-1}$.

c). Let $k \in \mathbb{N}$ with $k \leq n$. Prove that the subset $\mathfrak{X} = \{A \in \mathfrak{P}_k(\{1, \ldots, n\}) \mid \text{if } a \in A,\ \text{then } a+1 \notin A\}$ of $\mathfrak{P}_k(\{1, \ldots, n\})$ has cardinality $\binom{n-k+1}{k}$.

d). Let $X = \{x_1, \ldots, x_{2n+1}\}$, $n \in \mathbb{N}$ be a set with $2n+1$ elements. For $k = 0, 1, \ldots, n$, let $\mathfrak{X}_k$ be the set of all those subsets of X of cardinality $\geq n+1$ which contain x_{n+k+1} and exactly n elements from $x_1, \ldots, x_{n+k}$, i.e. $\mathfrak{X}_k = \{A \in \mathfrak{P}_{\geq n+1}(X) \mid |A \cap \{x_1, \ldots x_{n+k}\}| = n \text{ and } x_{n+k+1} \in A\}$. Show that $\bigcup_{k=0}^n \mathfrak{X}_k = \mathfrak{P}_{\geq n+1}(X)$ and hence deduce that $\sum_{k=0}^n 2^{n-k}\binom{n+k}{k} = 4^n$. (Note that subsets of X which are elements of $\mathfrak{X}_k$ may contain some elements from $x_{n+k+2}, \ldots, x_{n+1}$. See also Exercise T1.3-b), T1.4-e) and i).)

On the other side one can see (simple) test-exercises.

Test-Exercises

T1.1. (Tower of Hanoi) The puzzle consists of n disks of decreasing diameter placed on a pole. There are two other poles. The problem is to move the entire pile to another pole by moving one disk at a time to any other pole, except that no disk may be placed on top of a smaller disk. Find a formula for the least number of moves needed to move n disks from one pole to another, and prove the formula by induction.

T1.2. (Indicator functions) Let I be a set. For a subset $J \in \mathfrak{P}(I)$, let $e_J : I \to \{0,1\}$ be the indicator function of J (with respect to I), i.e. $e_J(i) = \begin{cases} 1, & \text{if } i \in J, \\ 0, & \text{if } i \in I \setminus J. \end{cases}$ Note that $e_I = 1$ and $e_\emptyset = 0$. Show that

a). The map $J \mapsto e_J$ is a bijective map from the poer set $\mathfrak{P}(I)$ onto the set $\{0,1\}^I$ of all maps $I \to \{0,1\}$.

b). For subsets $J, K \subseteq I$, prove that: $e_{J\cap K} = e_J e_K$, $e_{J\cup K} = e_J + e_K - e_J e_K$, $e_{J\setminus K} = e_J(1-e_K)$. In particular, $e_{I\setminus J} = 1 - e_J$ and $e_{J\,\triangle\, K} = e_J + e_K - 2e_J e_K$.

c). For $J, K \in \mathfrak{P}(I)$, let $J + K := J\Delta K := (J \cup K) \setminus (J \cap K)$ denote the symmetric difference of J and K. Show that: 1) $J + K = K + J$ and $J + \emptyset = J$, $J + J = \emptyset$. 2) $(J+K)+L = J+(K+L)$ for all $J, K, L \in \mathfrak{P}(I)$. 3) For every $J, L \in \mathfrak{P}(I)$, there exists a unique K such that $J + K = L$.
4) $(J+K)\cap L = (J\cap L)+(K\cap L)$ for all $J, K, L \in \mathfrak{P}(I)$.

(**Remark:** For verification of these properties use indicator functions and their rules given in b). These properties of the symmetric difference $\triangle$ show that the power set $\mathfrak{P}(I)$ with the symmetric difference Δ as addition and the intersection $\cap$ as multiplication is a commutative ring with $\emptyset$ as the zero element 0 and I as the unit element 1. This ring $(\mathfrak{P}(I), \triangle, \cap)$ is called the set-ring of I. Moreover, it is a finite dimensional algebra over the prime field $\mathbb{Z}_2$ of dimension $|I|$; if $|I| = 1$, then it is the prime field $\mathbb{Z}_2$; in the case $|I| > 1$, it is not a field – nor even an integral domain.)

T1.3. Use induction to prove that: for all $n \in \mathbb{N}$: **a).** $\sum_{k=0}^{n} k\cdot(k!) = (n+1)! - 1$. **b).** $\sum_{k=0}^{n} 2^{n-k}\binom{n+k}{k} = 4^n$. (see also Exercise 1.4-d).) **c).** $\sum_{k=m}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, $m \in \mathbb{N}$, $m \le n$.

T1.4. Let X be a finite set with n elements.

a). The number of subsets of X is 2^n. (**Hint:** The map $\mathfrak{P}(X) \to \{0,1\}^X$, $A \mapsto e_A$ is bijective.)

b). If $n \in \mathbb{N}^*$, then the number of subsets of X with an even number of elements is equal to the number of subsets of X with an odd number of elements. Moreover, this number is equal to 2^{n-1}. (**Hint:** Let $a \in X$. The map defined by $A \mapsto A \cup \{a\}$, if $a \notin A$, resp. $A \smallsetminus \{a\}$, if $a \in A$, is a bijective map from the set of subsets with an even number of elements onto the set of subsets with an odd number of elements.)

c). For $n \in \mathbb{N}$, prove that: $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n} = 2^n$. (**Hint:** Use part a).)

d). For $n \in \mathbb{N}^*$, prove that: $\binom{n}{0}-\binom{n}{1}+\cdots+(-1)^n\binom{n}{n} = 0$. (**Hint:** Use part b) or $(1-1)^n = 0$.)

e). Prove that $\sum_{k=0}^{n}\binom{2n+1}{2k} = 4^n = \sum_{k=0}^{n}\binom{2n+1}{2k+1}$ for $n \in \mathbb{N}$ and $\sum_{k=0}^{n}\binom{2n}{2k} = \frac{4^n}{2} = \sum_{k=0}^{n-1}\binom{2n}{2k+1}$ for $n \in \mathbb{N}^*$. (**Hint:** Use part b).)

f). Let Y be a k-element subset of X. Then the number of m-element subsets of X which contain Y is $\binom{n-k}{m-k}$.

g). For natural numbers m, n with $m \le n$, show that $\sum_{k=0}^{m}\binom{n}{k}\binom{n-k}{m-k} = 2^m\binom{n}{m}$. (**Hint:** Compute the sum of all numbers in the part e), where Y runs through all k-element subsets of X in two different ways or use the formula $\binom{n}{k}\binom{n-k}{m-k} = \binom{n}{m}\binom{m}{k}$.)

h). For $m, n, k \in \mathbb{N}$, $\binom{m+n}{k} = \binom{m}{0}\binom{n}{k}+\binom{m}{1}\binom{n}{k-1}+\cdots+\binom{m}{k}\binom{n}{0}$. In particular, $\binom{2n}{n} = \binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2$ for $n \in \mathbb{N}$. (**Hint:** Let X, Y be disjoint sets with $|X| = m$, $|Y| = n$. The assignment $A \mapsto (A\cap X,\ A\cap Y)$ defines a bijective map $\mathfrak{P}(X\cup Y) \to \mathfrak{P}(X)\times\mathfrak{P}(Y)$.)

i). What is the cardinality of the set $\mathfrak{P}_{\ge n+1}(\{1,2,\ldots,2n+1\})$? (see also Exercises T1.3-b), T1.4-e) and 1.4-d).)

T1.5. Let X be a finite set with n elements.

a). The number of pairs $(X_1,\ X_2)$ in $\mathfrak{P}(X)\times\mathfrak{P}(X)$ with $X_1 \cap X_2 = \emptyset$ is 3^n. More generally: The number of m–tuples $(X_1,\ldots,X_m)$ of pairwise disjoint subsets $X_1,\ldots,X_m \subseteq X$ is equal $(m+1)^n$.

b). For $n, r \in \mathbb{N}$, prove that $\sum_m \binom{n}{m} = r^n$, where m run through the set of r-tuples $(m_1,\ldots,m_r) \in \mathbb{N}^r$ of natural numbers with $m_1+\cdots+m_r = m$. (**Hint:** Use $r^n = (1+\cdots+1)^n$ or the part a).)