E0 219 Linear Algebra and Applications / August-December 2011

(ME, MSc. Ph. D. Programmes)

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212) **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Jasine Babu** (jasinekb@gmail.com) / **Nitin Singh** (nitin@math.iisc.ernet.in) / **Amulya Ratna Swain** (amulya@csa.iisc.ernet.in) / **Achintya Kundu** (achintya.ece@gmail.com)

1-st Midterm : Saturday, September 17, 2011; 15:00 -17:00 **2-nd Midterm :** Saturday, October 22, 2011; 10:30 -12:30
Final Examination : December ??, 2011, 10:00 -13:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

2. Vector Spaces

Submit a solution of the $*$-Exercise ONLY

Due Date : Monday, 22-08-2011 (Before the Class)

2.1 Let K be a field and let I be an index set.

(a) The set of all functions $f: I \to K$ with finite image i.e. $f(I)$ is a finite subset of K, is a K-subspace of the vector space K^I.

(b) The set of all functions $f: I \to K$ with countable image i.e. $f(I)$ is a countable subset of K, is a K-subspace of the vector space K^I.

(c) The set $\mathrm{B}_{\mathbb{K}}(I)$ of all bounded functions $f: I \to \mathbb{K}$ is a $\mathbb{K}$-subspace of $\mathbb{K}^I$.

(d) The set W_{even} (resp. W_{odd}) of all even (resp. odd) functions[1] $\mathbb{R} \to \mathbb{K}$ is a $\mathbb{K}$-subspaces of $\mathbb{K}^{\mathbb{R}}$. Further, show that $W_{\text{even}} \cap W_{\text{odd}} = 0$ and $W_{\text{even}} + W_{\text{odd}} = \mathbb{K}^{\mathbb{R}}$.

(e) The set of all functions $f: \mathbb{C} \to \mathbb{C}$ with $\lim_{z \to \infty} f(z) = 0$ is a $\mathbb{C}$-subspace of the vector space $\mathbb{C}^{\mathbb{C}}$ of all $\mathbb{C}$-valued functions on $\mathbb{C}$.

***2.2** Let V be a vector space over a field K with a field with $|K| \geq n$ and let $V_1, \ldots, V_n$ be K-subspaces of V. If $V_i \neq V$ for every $1 \leq i \leq n$ then show that $V_1 \cup V_2 \cup \cdots \cup V_n \neq V$. Show by an example that the condition $|K| \geq n$ is necessary. (**Hint :** By induction on n, assume that $V_1 \cup V_2 \cup \cdots \cup V_{n-1} \neq V$. Choose $x \in V_n$ with $x \notin V_1 \cup \cdots \cup V_{n-1}$ and $y \in V$ with $y \notin V_n$. Now consider the set $\{ax + y \mid a \in K\}$ which has at least n distinct elements.)

2.3 For subspaces U, U', W, W' of a vector space V over a field K, show that :

(a) The subset $V \setminus (U \setminus W)$ is a subspace of V if and only if $U = V$ or $U \subseteq W$.

(b) $U + (U' \cap W) \subseteq (U + U') \cap (U + W)$.

(c) $U \cap (U' + W) \supseteq (U \cap U') + (U \cap W)$.

(d) (Modular law) If $U \subseteq U'$, then $U + (U' \cap W) = U' \cap (U + W)$.

(e) Suppose that $U \cap W = U' \cap W'$. Then $U = \big(U + (W \cap U')\big) \cap \big(U + (W \cap W')\big)$.

2.4 Let K be a field and let $K[X]$ be the set of polynomials with coefficients in K. Let Φ denote the (evaluation) map $\Phi : K[X] \to K^K$ defined by $F(X) \mapsto (a \mapsto F(a))$. Show that

(a) Φ is injective if and only if K is not finite. (**Hint :** Use T2.6-(d).)

(b) Φ is surjective if and only if K is finite.(**Hint :** Remember *Polynomial interpolation*! See T2.8)

On the other side one can see auxiliary results and (simple) test-exercises.

[1] A function $f: \mathbb{R} \to \mathbb{K}$ is called even (respectively, odd) if $f(-x) = f(x)$ (respectively, $f(-x) = -f(x)$) for all $x \in \mathbb{R}$. For example, the sine $\sin: \mathbb{R} \to \mathbb{R}$ (respectively, cosine $\cos; \mathbb{R} \to \mathbb{R}$) function is an odd (respectively, even) function.

Auxiliary Results/Test-Exercises

T2.1 Let V be a vector space over a field K.

(a) (General Distributive law) For arbitrary finite families a_i, $i \in I$, in K and x_j, $j \in J$, in V, show that
$$\Big(\sum_{i\in I} a_i\Big)\Big(\sum_{j\in J} x_j\Big) = \sum_{(i,j)\in I\times J} a_i x_j\,.$$

(b) (Sign Rules) For arbitrary elements $a, b \in K$ and arbitrary vectors $x, y \in V$. Prove that :
(1) $0 \cdot x = a \cdot 0 = 0$. (2) $a(-x) = (-a)x = -(ax)$. (3) $(-a)(-x) = ax$.
(4) $a(x-y) = ax - ay$ and $(a-b)x = ax - bx$.

(c) (Cancelation Rule) Let $a \in K$ and let $x \in V$. If $ax = 0$ then $a = 0$ or $x = 0$.

T2.2 Let V be a vector space over a field and let X be any set with a bijection $f : X \to V$. Then X has a K-vector space structure with $f^{-1}(0)$ as a zero element and for $a \in K$, $x, y \in X$, $x + y := f^{-1}\big(f(x) + f(y)\big)$ and $ax := f^{-1}\big(af(x)\big)$.

T2.3 Let X be any set. Then the set-ring $(\mathfrak{P}(X), \Delta, \cap)$ of X (see exercise 1.5) has a natural structure of a vector space over the field $\mathbb{Z}_2$. (**Hint :** The map $\mathfrak{P}(X) \to \mathbb{Z}_2^X$ defined by $A \mapsto e_A$ is a bijective, where e_A denote the indicator function of A. See Test-Exercise T1.5.)

T2.4 Recall the concepts *convergent sequence*, *null- sequence*, *Cauchy sequence*, *bounded sequence* and *limit point of a sequence*.[2]

(a) Let $(\mathbb{R}^{\mathbb{N}})_{\rm conv}$ (respectively, $(\mathbb{R}^{\mathbb{N}})_{\rm null}$, $(\mathbb{R}^{\mathbb{N}})_{\rm Cauchy}$, $(\mathbb{R}^{\mathbb{N}})_{\rm bdd}$, $(\mathbb{R}^{\mathbb{N}})_{\rm lpt}$, $(\mathbb{R}^{\mathbb{N}})_{\rm const}$) denote the set of all convergent (respectively, null-sequences, Cauchy sequences, bounded sequences, sequences with exactly one limit point). Which of these are subspaces of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers?

(b) Verify the inclusions and equalities in the following diagram :
$$\begin{array}{ccccccccc}
\mathbb{R}^{\mathbb{N}} & \supseteq & (\mathbb{R}^{\mathbb{N}})_{\rm bdd} & & & & & & \\
\cup\!| & & \cup\!| & & & & & & \\
(\mathbb{R}^{\mathbb{N}})_{\rm lpt} & \supseteq & (\mathbb{R}^{\mathbb{N}})_{\rm lpt}\cap(\mathbb{R}^{\mathbb{N}})_{\rm bdd} & = & (\mathbb{R}^{\mathbb{N}})_{\rm Cauchy} & = & (\mathbb{R}^{\mathbb{N}})_{\rm conv} & \supseteq & (\mathbb{R}^{\mathbb{N}})_{\rm const} \\
 & & & & & & \cup\!| & & \\
 & & & & & & (\mathbb{R}^{\mathbb{N}})_{\rm null} & &
\end{array}$$

†**T2.5** (Function Spaces) Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $D \subseteq \mathbb{K}$ be an arbitrary subset.

(a) The set
$$C^0_{\mathbb{K}}(D) := \{f : D \to \mathbb{K} \mid f \text{ is continuous}\}$$
of all $\mathbb{K}$-valued continuous functions on D is a $\mathbb{K}$-subspace of all $\mathbb{K}$-valued functions $\mathbb{K}^D$ on D.

(b) Let $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$ with more than one point and let $n \in \mathbb{N}$. The set
$$C^n_{\mathbb{K}}(I) := \{f : I \to \mathbb{K} \mid f \text{ is } n-\text{timescontinuously differnetiable}\}$$

[2] A sequence $(x_n) = (x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ is called convergent (in $\mathbb{K}$) if there exists an element $x \in \mathbb{K}$ which satisfy the following property : For every positive (however small) real number $\varepsilon \in \mathbb{R}$ there exists a natural number $n_0 \in \mathbb{N}$ such that $|x_n - x| \leq \varepsilon$ for all natural numbers $n \geq n_0$. This element x is uniquely determined by the sequence (x_n) and is called the limit of the sequence (x_n) ; usually denoted by $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_n$. If x is the limit of (x_n), then this is also shortly written as $x_n \to x$ or $x_n \xrightarrow[n\to\infty]{} x$ and say that (x_n) converges to x. The sequence (x_n) converges to x if and only if the sequence $(x_n - x)$ converges to 0. A convergent sequence with limit 0 is called a null-sequence. A sequence that is not convergent is called divergent.

A sequence $(x_n) = (x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ is called bounded sequence if there exists an element S in $\mathbb{R}$ such that $|x_n| \leq S$ for all $n \in \mathbb{N}$.

A sequence $(x_n) = (x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ is called a Cauchy sequence if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ $|x_m - x_n| \leq \varepsilon$ for all natural numbers $m, n \geq n_0$.

An element $x \in \mathbb{K}$ is called a limit point of the sequence $(x_n) = (x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ if it is a limit point of the set $\{x_n \mid n \in \mathbb{N}\}$, i.e. every (however small) neighbourbood of x contain infinitely many terms of the sequence.

of all $\mathbb{K}$-valued n-times continuously differentiable functions on I is a $\mathbb{K}$-subspace the $\mathbb{K}$-vector space $C^0_{\mathbb{K}}(D)$.

(c) The $\mathbb{K}$-subspaces $C^n_{\mathbb{K}}(I)$, $n \in \mathbb{N}$ form a descending chain

$$C^0_{\mathbb{K}}(I) \supsetneqq C^1_{\mathbb{K}}(I) \supsetneqq C^2_{\mathbb{K}}(I) \supsetneqq \cdots \supsetneqq C^n_{\mathbb{K}}(I) \supsetneqq C^{n+1}_{\mathbb{K}}(I) \supsetneqq \cdots$$

where all inclusions are proper. The intersection of these K-subspaces is the K-subspace

$$C^\infty_{\mathbb{K}}(I) = \bigcap_{n\in\mathbb{N}} C^n_{\mathbb{K}}(I)$$

of all infinitely many times differentiable $\mathbb{K}$-valued functions on I.

(d) The set

$$C^\omega_{\mathbb{K}}(I) := \{f : I \to \mathbb{K} \mid f \text{ is analytic}\}$$

of all $\mathbb{K}$-valued analytic functions on I is a $\mathbb{K}$-subspace the $\mathbb{K}$-vector space $C^\infty_{\mathbb{K}}(I)$. Moreover, the inclusion $C^\omega_{\mathbb{K}}(I) \subsetneqq C^\infty_{\mathbb{K}}(I)$ is proper. (This follows from the existence of a "*flat functions*")

(e) Let $I \subseteq \mathbb{R}$ be an interval with more than one point and let $a_0, \ldots, a_{n-1}$ be complex valued continuous functions on I. The set of all functions $y \in C^n_{\mathbb{C}}(I)$ satisfying the (homogeneous linear) differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0y = 0$$

is a $\mathbb{C}$-subspace of $C^n_{\mathbb{C}}(I)$.

T2.6 (P o l y n o m i a l s – P o l y n o m i a l r i n g) A polynomial (in one variable or indeterminate X) with coefficients in a commutative ring A is a formal expression of the form: $F = F(X) = a_0 + a_1X + \cdots + a_nX^n$, where $n \in \mathbb{N}$ and the c o e f f i c i e n t s $a_0, a_1, \ldots, a_n \in A$. If $G = b_0 + b_1X + \cdots + b_mX^m$ is another polynomial, then $F = G$ if and only if $a_i = b_i$ for all $i \in \mathbb{N}$, where we put $a_i = 0$ for all $i > n$ and $b_j = 0$ for all $j > m$. The set of all polynomials with coefficients in a (given) ring A is denoted by $A[X]$, i. e.

$$A[X] := \{a_0 + a_1X + \cdots + a_nX^n \mid n \in \mathbb{N}, \quad a_0, \ldots, a_n \in A\}.$$

One can use addition, multiplication and distributive laws in the ring A to define addition and multiplication of polynomials:

$F + G := (a_0 + b_0) + (a_1 + b_1)X + \cdots$ and $F \cdot G := (a_0b_0) + (a_0b_1 + a_1b_0)X + (a_0b_2 + a_1b_1 + a_2b_0)X^2 + \cdots$

The i-th coefficient of the polynomial $F + G$ (respectively, $F \cdot G$) is $a_i + b_i$ (respectively, $a_0b_i + a_1b_{i-1} + \cdots + a_ib_0 = \sum_{j=0}^{i} a_jb_{i-j}$). With these addition and multiplication $A[X]$ is again a commutative ring with identity $1_{A[X]} = 1_A$; this ring is called the p o l y n o m i a l r i n g i n o n e i n d e t e r m i n a t e X o v e r A.

Let $F = \sum_{i=0}^{n} a_iX^i \in A[X]$ be a non-zero polynomial over a commutative ring A. The biggest natural number $n \in \mathbb{N}$ with $a_n \neq 0$ is called the d e g r e e of F and is denoted by $\deg F$. The corresponding coefficient a_n is called the l e a d i n g c o e f f i c i e n t of F. A polynomial with leading coefficient 1 is called a m o n i c polynomial. For $F \in A[X]$, if $\deg F = 0$, then $F = a \in A$, $a \neq 0$ is a non-zero constant polynomial. Below we record some computational rules for the degrees of polynomials:

(a) Let $F, G \in A[X]$ be non-zero polynomials. Then:

(1) $\deg(FG) \begin{cases} \leq \max\{\deg F, \deg G\}, & \text{if} \quad F + G \neq 0, \\ = \max\{\deg F, \deg G\}, & \text{if} \quad F + G = 0. \end{cases}$

(2) $\deg(FG) \begin{cases} \leq \deg F + \deg G & \text{if} \quad FG \neq 0 \\ = \deg F + \deg G & \text{if one of the leading coefficient of } F \text{ or } G \text{ is a } \textit{non-zero divisor} \text{ in } A. \end{cases}$

(Recall that an element $a \in A$ in a (commutative) ring A is called a z e r o d i v i s o r if there exists $b \in A$, $b \neq 0$ with $ab = 0$; An element which is not a zero divisor is called a n o n - z e r o d i v i s o r in A. For example, in the ring $\mathbb{Z}_n$ residue classes of divisors of n are precisely zero divisors. In the ring of integers $\mathbb{Z}$ every non-zero element is a non-zero divisor. In a field every non-zero element is a non-zero divisor. More generally, every invertible element in any ring is a non-zero divisor. A commutative ring which does not have any non-zero zero divisors is called an i n t e g r a l d o m a i n. For example, the ring of integers $\mathbb{Z}$ is an integral domain and every field K is an integral domain.)

(3) If A is an integral domain, then the invertible elements in the polynomial ring $A[X]$ are precisely the invertible elements in A, i. e. $A[X]^\times = A^\times$. In particular, a non-zero polynomial $F \in K[X]$ over a field K is

invertible in $K[X]$ if and only if it is a non-zero constant polynomial. In particular, X is never an invertible element in $K[X]$ and hence $K[X]$ is never a field.

(b) (Division algorithm for polynomials) Let F and $G \neq 0$ be polynomials over a field K. Then there exist unique polynomials Q and R over K such that

$$F = QG + R \quad \text{and} \quad \deg R < \deg G.$$

In particular, if $a \in K$, then $F = F(a) + Q(X-a)$, where Q is a polynomial over K. (**Remark :** More generally, one can perform division with remainder over arbitrary commutative ring by the polynomial G with an invertible leading coefficient.)

(c) (Zeros of polynomials) Let K be a field. An element $a \in K$ is called a zero of the polynomial $F \in K[X]$ if $F(a) = 0$. Therefore $a \in K$ is a zero of F if and only if $X - a$ divide F (in $K[X]$), i. e. $X - a$ is a linear factor of F.

(1) Let $F \in K[X]$ be a non-zero polynomial over a field K. Then there exist distinct elements $a_1, \ldots, a_r \in K$, $r \geq 0$, non-zero natural numbers $n_1, \ldots, n_r \in \mathbb{N}^+$ and a polynomial $G \in K[X]$ which does not have a zero in K, i. e. $G(a) \neq 0$ for every $a \in K$, such that

$$F(X) = (X - a_1)^{n_1} \cdots (X - a_r)^{n_r} \cdot G.$$

Moreover, the factors $(X - a_i)^{n_i}$, $i = 1, \ldots, r$ and G are uniquely (up to a permutation) determined by F. The elements $a_1, \ldots, a_r$ are (distinct) all zeros of F in K and the exponents $n_1, \ldots, n_r$ are called their multiplicities (or orders). The sum $n_1 + \cdots + n_r$ is the number of zeros of F in K counted with multiplicities. Naturally, $n_1 + \cdots + n_r + \deg G = \deg F$. In particular:

(2) Every polynomial F of degree $n \geq 0$ over a field K has at most n zeros in K (even if we count them with multiplicities). How many zeros the polynomial $X^2 + X$ has in the ring $\mathbb{Z}_4$? The polynomial $X^3 + X^2 + X + 1$ in $\mathbb{Z}_4[X]$ is a multiple of $X + 1$ and $X + 3$, but not of $(X+1)(X+3)$. Give an example of a polynomial $F \in A[X]$ over a commutative ring A such that F has infinitely many zeros in A.

(3) In the case $K = \mathbb{R}$ in general the polynomial G in (1) above can have positive degree. For example, the polynomial $X^2 + 1$ and its power have no zero in $\mathbb{R}$. However, a polynomial $F \in \mathbb{R}[X]$ of odd degree has at least one zero in $\mathbb{R}$, since $f(x) < 0$ (respectively, $f(x) > 0$) for large negative (respectively, positive) x.

(d) (Identity Theorem) Let $F, G \in K[X]$ be two polynomials with coefficients in K of degrees $\leq n$. Suppose that there exist distinct $t_1, \ldots, t_{n+1} \in K$ such that $F(t_i) = G(t_i)$ for all $i = 1, \ldots, n+1$. Then $F = G$. (**Hint :** Since $t_1, \ldots, t_{n+1} \in K$ are zeros of the polynomial $F - G$ of degree $\deg(F - G) \leq n$, it follows that $F - G = 0$ by (c) (2).)

T2.7 (Horner's scheme) Let K be a field and let $F = a_0 + a_1 X + \cdots + a_n X^n \in K[X]$. To compute the value of F at a point a one can apply the well-known Horner's scheme. For this define a sequence of polynomials recursively as follows :

$$\begin{aligned}
F_0 &:= a_n \\
F_1 &:= a_{n-1} + XF_0 = a_{n-1} + a_n X \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
F_{k+1} &:= a_{n-k-1} + F_k X = a_{n-k-1} + \cdots + a_{n-1} X^k + a_n X^{k+1} \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
F_n &:= a_0 + F_{n-1} X = F\,.
\end{aligned}$$

These polynomials are called the Ruffini's polynomials corresponding to F. The value $F(a) = F_n(a)$ is then obtained by the recursion-scheme:

$$F_0(a) = a_n\,, \qquad F_{k+1}(a) = a_{n-k-1} + F_k(a)a\,, \quad k = 0, \ldots, n-1$$

The values $F_0(a), \ldots, F_n(a)$ can be easily computed one after the another and the division algorithm by $X - a$ is given by

$$F = Q \cdot (X - a) + F(a) \quad \text{where} \quad Q = F_0(a)X^{n-1} + F_1(a)X^{n-2} + \cdots + F_{n-1}(a)\,,\ F(a) = F_n(a)\,.$$

With this process also one can easily compute all coefficients b_ν in the *Taylor's expansion* :

$$F = b_0 + b_1(X-a) + \cdots + b_n(X-a)^n\,,\ \ b_k = F^{(k)}(a)/k!\,,$$

for this one has to repeat the above process for the polynomial Q instead of F and hence $b_1 = Q(a)$, and so on. For example, the polynomial $F = 2X^3 + 2X^2 - X + 1$ and $a = -2$ we have the following scheme :

	2	2	−1	1
−2	2	−2	3	$-5(=b_0)$
−2	2	−6	$15(=b_1)$	
−2	2	$-10(=b_2)$		
−2	$2(=b_3)$			

Therefore $F = 2(X+2)^3 - 10(X+2)^2 + 15(X+2) - 5$.

T2.8 (Polynomial interpolation) Let K be a field and let $m \in \mathbb{N}$. The existence of a polynomial $f \in K[X]$ of degree $\leq m$ which has given $m+1$ values (in K) at distinct $m+1$ places is called an *interpolation problem*.

(a) (Lagrange's interpolation formula) Let $a_0, \ldots, a_m \in K$ be distinct and let $b_0, \ldots, b_m \in K$ be given. Then

$$f := \sum_{i=0}^{m} \frac{b_i}{c_i} \prod_{j \in \{0,\ldots,m\} \setminus \{i\}} (X - a_j)\,,\ c_i := \prod_{j \in \{0,\ldots,m\} \setminus \{i\}} (a_i - a_j)$$

is the unique polynomial (by T2.6-(d)) of degree $\leq m$ such that $f(a_i) = b_i$ for all $i = 0, \ldots, m$.

(b) (Newton's interpolation) Let $f_0 := 1, f_1 := X - a_0, f_2 := (X-a_0)(X-a_1), \ldots, f_m := (X-a_0)\cdots(X-a_{m-1})$. Then, since $f_j(a_j) \neq 0$, we can recursively find the coefficients $\alpha_0, \ldots, \alpha_m \in K$ such that

$$\left(\sum_{j=0}^{r} \alpha_j f_j\right)(a_r) = b_r\,,\ 0 \leq r \leq m\,.$$

The polynomials $\sum_{j=0}^{r} \alpha_j f_j$ have degree $\leq r$ and values b_i at the points a_i for all $i = 0, \ldots, m$.

T2.9 (Polynomial functions) Let K be a field and let $D \subseteq K$ be a subset of K. A function $f : D \to K$ is called a *polynomial function* if it is of the form $t \mapsto a_0 + a_1 t + \cdots + a_n t^n$ with fixed *coefficients* $a_0, a_1, \ldots, a_n \in K$.

(a) The set of all polynomial functions $\mathrm{Pol}_K(D)$ form a K-subspace of the K-vector space K^D. Moreover, if $K = \mathbb{K}$ and if $D = I \subseteq \mathbb{R}$ is an interval with more than one point, then $\mathrm{Pol}_{\mathbb{K}}(I) \subseteq \mathrm{C}^{\omega}_{\mathbb{K}}(I)$.

(b) If D is a finite subset of K, then every K-valued function on D is a polynomials function, i. e. $K^D = \mathrm{Pol}_K(D)$.

(c) If D is an infinite set, then the coefficients $a_0, a_1, \ldots, a_n \in K$ of the polynomial function $f : D \to K$, $t \mapsto a_0 + a_1 t + \cdots + a_n t^n$ are uniquely determined by the function f.

(d) The functions $\mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$; $x \mapsto \sin x$; $x \mapsto \cos x$ are not polynomial functions. Is the exponential function $x \mapsto e^x$ a polynomial function?

T2.10 (Rational functions) Let K be a field. The quotient of two polynomials over K are called the *rational functions (in one variable X over K)*. Therefore a rational function is of the form F/G with $F, G \in K[X]$. The set of all rational functions is denoted by $K(X)$.

(a) Sum and product of rational functions are again rational functions and so $K(X)$ is a vector space over K and $K[X]$ is a K-subspace of $K(X)$. Further, $K(X)$ is a field and is called the *rational function field (in one variable X over K)*.

(b) Every rational function F/G in one indeterminate X over K can also be represented as $F/G = Q + R/G$, where Q and R are polynomials over K with $\deg R < \deg G$.

(c) (Partial fraction decomposition) Let F and G be polynomials over K with $\deg F < \deg G$ and $G = (X-\alpha_1)^{n_1} \cdots (X-\alpha_r)^{n_r}$, $\alpha_i \neq \alpha_j$ for $i \neq j$, $n_i \in \mathbb{N}^*$. Then there exists a unique representation

$$\frac{F}{G} = \frac{\alpha_{11}}{(X-\alpha_1)} + \frac{\alpha_{12}}{(X-\alpha_1)^2} + \cdots + \frac{\alpha_{1n_1}}{(X-\alpha_1)^{n_1}} + \cdots\cdots + \frac{\alpha_{r1}}{(X-\alpha_r)} + \frac{\alpha_{r2}}{(X-\alpha_r)^2} + \cdots + \frac{\alpha_{rn_r}}{(X-\alpha_r)^{n_r}}\,.$$

with $\alpha_{ik} \in K$, $i = 1, \ldots, r$; $k = 1, \ldots, n_i$.