E0 219 Linear Algebra and Applications / August-December 2011

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

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1-st Midterm : Saturday, September 17, 2011; 10:30 -12:30 **2-nd Midterm :** Saturday, October 22, 2011; 10:30 -12:30
Final Examination : December ??, 2011, 10:00 -13:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

4. Dimension of vector spaces

Submit a solution of the $*$-Exercise ONLY

Due Date : Monday, 05-09-2011 (Before the Class)

Let K be arbitrary field and let $\mathbb{K}$ denote either the field $\mathbb{R}$ or the field $\mathbb{C}$.

4.1 Let $\omega \in \mathbb{R}_+^\times$ be a fixed positive real number. For $a \in \mathbb{R}$ and $\varphi \in \mathbb{R}$, let $f_{a,\varphi} : \mathbb{R} \to \mathbb{R}$ be the function defined by $t \mapsto a\sin(\omega t + \varphi)$ and let $W := \{f_{a,\varphi} \mid a, \varphi \in \mathbb{R}\}$. Then W is a $\mathbb{R}$-subspace of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$ of all $\mathbb{R}$-valued functions on $\mathbb{R}$.

(a) Find a $\mathbb{R}$-basis of the $\mathbb{R}$-subspace W. What is the dimension $\mathrm{Dim}_{\mathbb{R}} W$? (**Hint :** The functions $t \mapsto \sin\omega t$ and $t \mapsto \cos\omega t = \sin(\omega t + \pi/2)$ form a basis of W. — **Remark:** Elements of W are called harmonic oscillations with the circular frequency ω.)

(b) Show that every $f \neq 0$ function in W has a unique representation

$$f(t) = a\sin(\omega t + \varphi)\,, \qquad a > 0 \quad \text{and} \quad 0 \le \varphi < 2\pi\,.$$

(**Remark :** This unique a is called the amplitude and φ is called the phase angle of f. The zero function has the amplitude 0 and an arbitrary phase angle.)

(c) From the amplitudes and the phase angles of two harmonic oscillations f and g, compute the amplitudes and the phase angles of the functions $f \pm g$.

4.2 Let V be a K-vector space of dimension $n \in \mathbb{N}$.

(a) If $H_1, \ldots, H_r$ are hyper-planes in V, then show that $\mathrm{Dim}_K(H_1 \cap \cdots \cap H_r) \ge n - r$.

(b) If $U \subseteq V$ is a subspace of codimension r, then show that there exist r hyper-planes $H_1, \ldots, H_r$ in V such that $U = H_1 \cap \cdots \cap H_r$.

***4.3** Let $x_1 = (a_{11}, \ldots, a_{1n}), \ldots, x_n = (a_{n1}, \ldots, a_{nn})$ be elements of $\mathbb{K}^n$ with

$$|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ji}| \quad \text{for all } i = 1, \ldots, n.$$

Show that $x_1, \ldots, x_n$ is a basis of $\mathbb{K}^n$. (**Hint :** It is enough to show the linear independence of $x_1, \ldots, x_n$. For this, suppose that $b_1 x_1 + \cdots + b_n x_n = 0$ with $|b_i| \le 1$ for all i and $b_{i_0} = 1$ for some i_0. This already contradicts the give condition for i_0.)

4.4 Let $x_1, \ldots, x_n \in \mathbb{Z}^n$ be arbitrary vectors with integer components. For every $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$, the vectors $x_1 + \lambda e_1, \ldots, x_n + \lambda e_n$ form a basis of $\mathbb{Q}^n$. (**Hint :** Suppose $a_1(x_1 + \lambda e_1) + \cdots + a_n(x_n + \lambda e_n) = 0$ with $a_1, \ldots, a_n \in \mathbb{Z}$ and $\gcd(a_1, \ldots, a_n) = 1$ and lead to contradict the condition $\gcd(a_1, \ldots, a_n) = 1$.)

4.5 Let K be a field with at least n elements $(n \in \mathbb{N}^*)$ and let V be a finite dimensional K-vector space. Let $U_1, \ldots, U_n$ be subspaces of V of equal dimension r and let $u_{1i}, \ldots, u_{ir}$ be a basis of U_i for $i = 1, \ldots, r$. Show that there exists $\mathrm{Dim}_K V - r$ vectors in V such that which simultaneously extend the given bases of U_i to a basis of V. (**Hint :** Use Exercise 2.2.)

On the other side one can see auxiliary results and (simple) test-exercises.

Auxiliary Results/Test-Exercises

To understand and appreciate the Test-Exercises which are marked with the symbol † one may possibly require more mathematical maturity than one has! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

T4.1 Compute the dimension of $U, W, U+W$ and $U\cap W$ for the following subspaces U, W of the given vector space V.

(a) $V := \mathbb{R}^3$, $U := \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1+x_2=0,\ -x_2+x_3=0\}$,
$W := \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1+x_3=0,\ x_1-x_2-x_3=0\}$.

(b) $V := \mathbb{R}^4$, $U := \{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_1-x_2+x_3=0,\ x_1+x_2-x_4=0\}$,
$W := \{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_1+x_2-3x_3=0,\ x_1+2x_3-x_4=0\}$.

(c) $V := \mathbb{R}^5$, $U := \mathbb{R}x_1+\mathbb{R}x_2+\mathbb{R}x_3$, $W := \mathbb{R}y_1+\mathbb{R}y_2$ mit $x_1 := (1,1,0,1,0)$, $x_2 := (0,1,1,0,1)$, $x_3 := (0,1,1,0,0)$, $y_1 := (0,0,1,1,0)$, $y_2 := (1,1,-1,0,-1)$.

T4.2 Let $n\in\mathbb{N}$, $n\geq 2$. Determine whether or not the vectors $(1,1,1,\ldots,1)$, $(1,2,1,\ldots,1)$, $\ldots$, $(1,\ldots,1,n)$ form a basis of $\mathbb{R}^n$.

T4.3 **(a)** Let $W\subseteq\mathbb{R}^4$ be the subspace generated by $y_1 := (1,2,3,4)$, $y_2 := (4,3,2,1)$, $y_3 := (-1,0,1,2)$, $y_4 := (0,1,0,1)$, $y_5 := (1,3,-2,0)$. List all bases of W which are the subsequences of $y_1,\ldots,y_5$.

(b) Let $U\subseteq\mathbb{R}^4$ be the subspace generated by the vectors $x_1 := (0,12,-3,10)$, $x_2 := (1,7,-3,2)$, $x_3 := (-1,5,0,7)$, $x_4 := (1,3,-2,-1)$ and let $W\subseteq\mathbb{R}^4$ be the subspace as in the part (a).
(1) From $x_1,\ldots,x_4$ choose a basis of U and extend it to a basis of $U+W$ by using the vectors $y_1,\ldots,y_5$.
(2) Give a basis of $U\cap W$.

T4.4 Compute the co-ordinates of the vectors

(a) $(\mathrm{i},0),(1+\mathrm{i},-2+3\mathrm{i}),(0,1)$ with respect to the basis $v_1=(1+\mathrm{i},\mathrm{i}), v_2=(1,1+\mathrm{i})$ of the $\mathbb{C}$-vector space $\mathbb{C}^2$.

(b) $(1,0,-5\mathrm{i}),(2+\mathrm{i},1,0)$ with respect to the basis $v_1=(1,0,1-\mathrm{i}), v_2=(2+\mathrm{i},-1,-\mathrm{i}), v_3=(0,1+\mathrm{i},2-\mathrm{i})$ of the $\mathbb{C}$-vector space $\mathbb{C}^3$.

T4.5 Let K be a field. For which $(a,b)\in K^2$, the vectors $(a,b),(b,a)$ for a basis of K^2.

T4.6 Show that the elements $x_1,\ldots,x_n$ of the K-vector space V are linearly independent if and only if the subspace $U := Kx_1+\cdots+Kx_n$ has dimension n.

T4.7 Let x_i, $i\in I$, be a family of vectors in a K-vector space V and let U be a subspace of V generated by x_i, $i\in I$. Show that U is finite dimensional if and only if there exists a natural number $n\in\mathbb{N}$ such that every $n+1$ vectors among x_i, $i\in I$, are linearly dependent. Moreover, if this condition is satisfied then the dimension $\mathrm{Dim}_K U$ is the minimum of the $n\in\mathbb{N}$ with this property.

T4.8 Let K be a finite field with q elements. Show that a K-vector space of dimension $n\in\mathbb{N}$ has exactly q^n elements.

T4.9 Let K be a finite field with q elements.

(a) The multiples $m\cdot 1_K$, $m\in\mathbb{Z}$, form a subfield K' of K.

(b) There exists a smallest positive natural number p such that $p\cdot 1_K=0$. Moreover, it is prime (and is called the Characteristic of K, The subfield $K'\subseteq K$ contains exactly p distinct elements $0,1_K,\ldots,(p-1)1_K$.

(c) Show that $q=p^n$ with $n := \mathrm{Dim}_{K'}K$.

(**Remark :** *Therefore the number of elements is a finite field is a power of a prime number.* Conversely, (we shall prove later that) for a given prime-power q there exists (essentially unique) field with q elements.)

T4.10 Let V be a finite dimensional K-vector space and let U be a subspace of V. Let $u_1,\ldots,u_m$ be a basis of U and let $u_1,\ldots,u_m,u_{m+1},\ldots,u_n$ be an extended basis of V. Show that

$$x = a_1u_1+\cdots+a_mu_m+b_{m+1}u_{m+1}+\cdots+b_nu_n\in V$$

is an element of U if and only if the coordinates $b_{m+1}=u^*_{m+1}(x),\ldots,b_n=u^*_n(x)$ of x with respect to the basis $u_1,\ldots,u_n$ of V are zero. (**Remark :** This is the most common method of characterizing the elements of a subspace.)

T4.11 Let V be a $\mathbb{C}$-vector space of dimension $n \in \mathbb{N}^*$ and let H be a real hyperplane in V (i. e. a real subspace of dimension $2n-1$). Then show that $H \cap iH$ is a complex hyper-plane in V (i. e. a complex subspace of dimension $n-1$), where we put $iH := \{ix \mid x \in H\}$.

T4.12 Let U_1, U_2, U_3 be finite dimensional subspaces of a K-vector space V with $U_i \cap U_j = 0$ for $i \neq j$. Show that $\mathrm{Dim}\big((U_1+U_2)\cap U_3\big) = \mathrm{Dim}\big((U_1 \cap (U_2+U_3)\big) = \mathrm{Dim}\,U_1 + \mathrm{Dim}\,U_2 + \mathrm{Dim}\,U_3 - \mathrm{Dim}\,(U_1+U_2+U_3)$.

T4.13 Let V be a K-vector space with a countably infinite basis. Show that for every subspace U of V there exists a countable basis. (**Hint :** Let x_i, $i \in \mathbb{N}$, be a basis of V and let $V_n := Kx_0 + \cdots + Kx_n$. Then $U = \bigcup_{n=0}^{\infty}(U \cap V_n)$.)

T4.14 Let U be the subspace generated by the following functions in a space of a;; real-valued functions on $\mathbb{R}$. Compute the dimension of U, by choosing a basis from the given generating system and expressing other functions in this generating system as the linear combinations of the basis chosen.

(a) t^2, $(t+1)^2$, $(t+2)^2$, $(t+3)^2$. **(b)** $\sinh 3t$, $\cosh 3t$, e^{3t}, e^{-3t}.

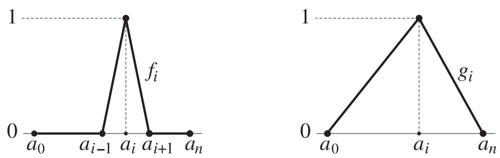
(c) 1, $\sin t$, $\sin 2t$, $\sin^2 t$, $\cos t$, $\cos 2t$, $\cos^2 t$. **(d)** 1, $\sinh t$, $\sinh 2t$, $\sinh^2 t$, $\cosh t$, $\cosh 2t$, $\cosh^2 t$.

T4.15 Let $n \in \mathbb{N}^*$ and let $a_0, \ldots, a_n$ be real numbers with $a_0 < a_1 < \cdots < a_n$.

(a) Let U be the $\mathbb{R}$-vector space of continuous piecewise linear [1] real valued functions os the closed interval $[a_0, a_n]$ in $\mathbb{R}$ with partition points $a_1, \ldots a_{n-1}$. Show that the functions $|t-a_0|, \ldots, |t-a_n|$ is a $\mathbb{R}$-basis of U. In particular, $\mathrm{Dim}_K U = n+1$.

(b) Let V be the $\mathbb{R}$-vector space of the continuous piecewise linear functions $\mathbb{R} \to \mathbb{R}$ with partition points $a_0, \ldots, a_n$. Show that the functions $(a_0-t)_+$, $|t-a_0|, \ldots, |t-a_n|$, $(t-a_n)_+$ is a basis of V, where $f_+ := \mathrm{Max}\,(f, 0)$ denote the positive part of a real valued function f. In particular, $\mathrm{Dim}_K V = n+3$.

(c) Let W be the $\mathbb{R}$-vector space of the continuous piecewise linear functions $[a_0, a_n] \to \mathbb{R}$ with partitions points $a_1, \ldots, a_{n-1}$, and which vanish at both the end points a_0 and a_n. Show that there exist functions $f_1, \ldots, f_{n-1} \in W$ and the functions $g_1, \ldots, g_{n-1} \in W$ which form bases of W such that the graphs of f_i and g_i are:



(d) Let $k, m \in \mathbb{N}$ with $k < m$. The set of k-times continuously differentiable $\mathbb{R}$-valued functions on the closed interval $[a_0, a_n]$, which are polynomial functions of degree $\leq m$ on every subinterval $[a_i, a_{i+1}]$, is a $\mathbb{R}$-vector space of dimension $(m-k)n + k + 1$ with basis

$$1,\ (t-a_0), \ldots, (t-a_0)^m, \big((t-a_1)_+\big)^{k+1}, \ldots, \big((t-a_1)_+\big)^m, \ldots, \big((t-a_{n-1})_+\big)^{k+1}, \ldots, \big((t-a_{n-1})_+\big)^m.$$

(**Remark :** The elements of this vector space are called spline functions of type (m,k) on $[a_0, a_n]$ with partition points $a_1, \ldots, a_{n-1}$.)

T4.16 For pairwise distinct elements $\lambda_0, \ldots, \lambda_n$ of a field K, in which the multiples $m \cdot 1_K$, $m \in \mathbb{N}^*$, are all $\neq 0$, [2] show that the polynomial functions $(t-\lambda_0)^n, \ldots, (t-\lambda_n)^n$ form a basis of the space $K[t]_{n+1}$ of all polynomial functions of degree $\leq n$ on K. (**Hint :** It is enough to prove the linear independence, do this by using binomial formula and Test-Exercise T3.6. – More generally, if $f(t) = \sum_{j=0}^{n} b_j t^j$ is an arbitrary polynomial function of degree n over K, then (under the given hypotheses on K and on $\lambda_0, \ldots, \lambda_n$) the polynomial functions $f(t-\lambda_0), \ldots, f(t-\lambda_n)$ are linearly independent. Note that the k-th derivative $f^{(k)}(t) = \sum_{j=k}^{n} j(j-1)\cdots(j-k+1) b_j t^{j-k}$ is a polynomial function of degree $n-k$ for every $k = 0, \ldots, n$ (This also holds for arbitrary field K of characteristic 0 or $> n$, if one use the

[1] Let $n \in \mathbb{N}^*$ and let $a_0, \ldots, a_n$ be real numbers with $a_0 < a_1 < \cdots < a_n$. A continuous real valued function $f : [a_0, a_n] \to \mathbb{R}$ is called piecewise linear with partition points $a_0, \ldots, a_n$ if $f\big|[a_i, a_{i+1}] \to \mathbb{R}$ is linear (see below) for every $i = 1, \ldots, n-1$. A real valued function $f : [a, b] \to \mathbb{R}$ defined on the closed interval $[a, b] \subseteq \mathbb{R}$ is called linear if there exist $\lambda, \mu \in \mathbb{R}$ such that $f(t) = \lambda t + \mu$ for every $t \in [a, b]$.

[2] In this one also says that K has the characteristic 0.

formal derivatives)[3] In particular, by Exercise 3.3-(a), the polynomial functions $f = f^{(0)}, f^{(1)}, \ldots, f^{(n)} = n! b_n$ form a basis of $K[t]_{n+1}$ and hence by using binomial formula and interchanging the summations, we get the well-known Taylor-Formula for f at $\lambda \in K$:

$$f(t-\lambda)=\sum_{j=0}^{n} b_j(t-\lambda)^j=\sum_{j=0}^{n} b_j \sum_{k=0}^{j} \binom{j}{k} t^{j-k}(-\lambda)^k=\sum_{k=0}^{n}(-1)^k\lambda^k \sum_{=k}^{n} j(j-1)\cdots(j-k+1)\, b_j \frac{t^{j-k}}{k!}=\sum_{k=0}^{n}(-1)^k\lambda^k \frac{f^{(k)}(t)}{k!}.$$

Now, to prove linear independence, consider $0=\sum_{i=0}^{n} a_i f(t-\lambda_i)$ with coefficients $a_i \in K$. From this it follows that $0=\sum_{i=0}^{n} a_i f(t-\lambda_i) = \sum_{i=0}^{n} a_i \sum_{k=0}^{n} (-1)^k \lambda_i^k \frac{f^{(k)}(t)}{k!} = \sum_{i=0}^{n} \frac{(-1)^k}{k!}\left(\sum_{i=0}^{n} a_i \lambda_i^k\right) f^{(k)}(t)$ and hence by the linear independence of $f = f^{(0)}, f^{(1)}, \ldots, f^{(n)}$ we have $\sum_{i=0}^{n} a_i \lambda_i^k = 0$ for all $k = 0, \ldots, n$. Now, use the Test-Exercise T3.6 to conclude that $a_0 = \cdots = a_n = 0$.)

†**T4.17** Let $n \in \mathbb{N}^*$. Show that there exist a representation in $\mathbb{Q}[t]$ of the form

$$t=\sum_{k=0}^{n}\frac{a_k}{b}(t+k)^n\,, \qquad a_k \in \mathbb{Z},\ b \in \mathbb{N}^*\,.$$

Use this to deduce that there exists a natural number $\gamma(n)$ such that every natural number is a sum of $\gamma(n)$ integers of the form $\pm m^n$, $m \in \mathbb{N}$. (**Hint :** For a representation use the above Test-Exercise T4.16. For multiples of b the assertion directly follows from the above formula, otherwise apply division with remainder. – **Remarks:** Further, one can choose $\gamma(n) \le |a_0| + \cdots + |a_n| + [b/2]$. In particular, one can even have $\gamma(2) = 3$ and $\gamma(3) = 5$, where it is still unknown whether or not $\gamma(3) = 4$. Since 6 and 14 can not be written in the form $m_1^2 \pm m_2^2$, the equality $\gamma(2) = 2$ is not enough. – The *Two-Square Theorem* (Fermat-Euler) describes exactly those natural numbers $m \in \mathbb{N}$ which can not be expressed in the form $m_1^2 \pm m_2^2$. Since 4 and 5 can not be expressed in the form $m_1^3 \pm m_2^3 \pm m_3^3$, as one sees this by computing modulo 9, it follows that the equality $\gamma(3) = 3$ is not enough. – Moreover, it is conjectured by E. Waring[4] (and von D. Hilbert proved it, even sharper) that: *There exists a natural number $g(n)$ such that every natural number is sum of $g(n)$ natural numbers of the form m^n, $m \in \mathbb{N}$*. In other words: *To determine , for a given positive natural number n, there is a natural number $g(n)$ such that the equation $a = x_1^n + \cdots x_{g(n)}^n$ has a solution in $\mathbb{N}^{g(n)}$ for every $a \in \mathbb{N}$*. This is known as the Waring's Problem. Previous writers had proved its existence when $n = 3, 4, 5, 6, 7, 8$ and 10, but its value $g(n)$ is determined only for $n = 3$. The value $g(n)$ is now known for all n. For example, $g(2) = 4$, $g(3) = 9$, $g(4) = 19$, $g(5) = 37$. Except for $g(2)$ and $g(3)$, the known proofs of these results involve much more complicated methods and use heavily the theory of functions of complex variable.)

T4.18 Let K be a field and let $c_0, \ldots, c_{n-1} \in K$ elements in K. Show that the subset $V \subseteq K^{\mathbb{N}}$ of all sequences (a_k) from $K^{\mathbb{N}}$, which satisfy the recursion-relation

$$a_{k+n} = c_0 a_k + c_1 a_{k+1} + \cdots + c_{n-1} a_{k+n-1}\,,$$

$k \in \mathbb{N}$, is a subspace of $K^{\mathbb{N}}$ of the dimension n. (**Hint:** If $K = \mathbb{C}$ and if

$$1 - c_{n-1}x - \cdots - c_0 x^n = (1-\beta_1 x)^{n_1} \cdots (1-\beta_r x)^{n_r}$$

holds in $\mathbb{C}[x]$ with pairwise distinct $\beta_1, \ldots, \beta_r \in \mathbb{C}$, then the sequences

$$(\beta_1^k), \ldots, (k^{n_1-1}\beta_1^k), \ldots, (\beta_r^k), \ldots, (k^{n_r-1}\beta_r^k)$$

form a $\mathbb{C}$-Basis of V. It is easy to see that these sequences belong to V and are linearly independent.)

†**T4.19** **(a)** Let $U \subseteq K^n$ be a subspace of dimension m. Then there exists uniquely determined basis of U of the form

$$v_1 = (*, \ldots, *, 1, 0, \ldots, 0) \in K^n\,,$$

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[3]**Formal derivatives** Let K be a field. For a polynomial $F = \sum_{n\in\mathbb{N}} a_n X^n \in K[X]$, we define the (formal) derivative of F by $F' := \sum_{n\in\mathbb{N}} n a_n X^{n-1} \in K[X]$. This formal derivative satisfies usual product and quotient rules: $(FG)' = F'G + FG'$ for all $F, G \in K[X]$ and $\left(\frac{F}{G}\right)' = \frac{GF' - G'F}{G^2}$ for all $F, G \in K[X], G \neq 0$.

[4]An English mathematician E. Waring stated without proof that every number is the sum of 4 squares, of 9 cubes, of 19 biquadrates, and so on in *Meditationes algebraicae* (1770), 204-205 and Lagrange proved that $g(2) = 4$ (*Lagrange's four-square theorem*) later in the same year. It is very improbable that Waring had any sufficient grounds for his assertion and it was until more than 100 years later that Hilbert first proved (even sharper assertion) that it is true. Hilbert's proof of the existence of $g(n)$ for every n was published in *Göttinger Nachrichten* (1909), 17-36 and *Math.Annalen*, **67** (1909), 281-305.

$$v_2 = (*,\ldots,*,0,*,\ldots,*,1,0,\ldots,0) \in K^n\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$v_m = (*,\ldots,*,0,*,\ldots,*,0,*,\ldots,*,0,\ldots,1,0,\ldots,0) \in K^n\,,$$
where at the position of $*$ there are elements in K which are uniquely determined by U and the positions d_j where there is 1 in the vectors v_j are uniquely determined by U, $1 \le d_1 < d_2 < \cdots < d_m \le n$. In the vectors v_j there are 0s at the positions $d_1,\ldots,d_{j-1}, d_j+1,\ldots,n$. (**Remarks:** The set of all m-dimensional subspaces of K^n is called the Grassmann-Mannifold $\mathrm{G}_K(m,n)$ of the type (m,n) over K. This Exercise gives a partition of $\mathrm{G}_K(m,n)$ into subsets $\sigma(d_1,\ldots,d_m)$, where $(d_1,\ldots,d_m)$ runs through the subsets of $\{1,2,\ldots,n\}$ of cardinality $\binom{n}{m}$ with $1 \le d_1 < \cdots < d_m \le n$. The subspace corresponding to $\sigma := \sigma(d_1,\ldots,d_m)$ is then parameterized by the tuple in K^{k_σ} where
$$k_\sigma := (d_1-1)+\cdots+(d_m-m) = \sum_{j=1}^{m} d_j - \binom{m+1}{2}\,.$$
$\sigma(d_1,\ldots,d_m)$ is called a Schubert-cell of the dimension
$$k_\sigma = \sum_{j=1}^{m} d_j - \binom{m+1}{2}$$
in $\mathrm{G}_K(m,n)$. Further, $\sigma(1,\ldots,m)$ respectively, $\sigma(n-m+1,\ldots,n)$ are the only Schubert-cells of the minimal dimension 0 respectively, the maximal dimension $m\ell$, $\ell := n-m$. – The definition of the Schubert-cells and their notation is not uniform in the literature. If we put $\delta_j := d_j - j$, $j=1,\ldots,m$, then a sequence $0 \le \delta_1 \le \cdots \le \delta_m \le \ell$ and the corresponding cell has the dimension $\delta_1+\cdots+\delta_m$. *For a given $k \in \mathbb{N}$, the number of Schubert-cells of dimension k is therefore equal to the number $p(k;m,\ell)$ of partitions of the number k with at most m positive natural numbers $\le \ell$.* For example, if K is a finite field with q elements, then
$$\mathrm{Card}\, G_K(m,n) = \sum_{k=0}^{m\ell} p(k;m,\ell)\, q^k\,.$$
Moreover, *this sum is equal to the value $G_m^{[n]}(q)$ of the Gauss-polynomial $G_m^{[n]}$ at the place q.* One can use this result and the Identity-Theorem for polynomials to give a combinatorial proof of the following equality of polynomials:
$$G_m^{[n]}(T) = \sum_{k=0}^{m\ell} p(k;m,\ell)\, T^k = \frac{(T^n-1)\cdots(T^{n-m+1}-1)}{(T^m-1)\cdots(T-1)}\,, \quad \ell = n-m\,.)$$

(b) Compute the bases described in part a) for the subspaces U and for W given in the Test-Exercise T4.3-(b) and T4.3-(a), respectively.

T4.20 Which numbers can occur as the dimensions of the intersections of a p-dimensional and a q-dimensional subspaces in a K-vector space of dimension n in Question?

T4.21 Let $V = Kx_1+\cdots+Kx_n+Kx_{n+1}$ be a K-vector space, W be a K-subspace of V with $W \not\subseteq V' := Kx_1+\cdots+Kx_n$ and let y be an arbitrary vector in $W \setminus V'$. Then show that
$$W = W \cap V' + Ky\,.$$
By induction on n it follows directly that every subspace of a K-vector space which a generating system consisting of n vectors, itself has a generating system consisting of at most n vectors.

T4.22 Let $v_1,\ldots,v_n$ be a basis of the n-dimensional K-vector space V, $n \ge 1$, and let H be a hyperplane in V. Show that there exist an index i_0, $1 \le i_0 \le n$, and elements $a_i \in K$, $i \ne i_0$ such that $v_i - a_i v_{i_0}$, $i \ne i_0$ is a basis of H. In which case for *every* $i_0 \in \{1,\ldots,n\}$ there are such elements $a_i \in K$?

T4.23 Let K be a field,, V be a n-dimensional K-vector space and
$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq V$$
be a sequence of K-subspaces with $\mathrm{Dim}_K V_i \le i$ for $i = 0,\ldots,n$. Then show that there is a flag
$$0 = W_0 \subset W_1 \subset \cdots \subset W_n = V$$
in V with $V_i \subseteq W_i$ for all $i = 1,\ldots,n$.

T4.24 Let V be a finite dimensional K-vector space and $m \in \mathbb{N}$. If V_i, $i \in I$, are subspaces of V with
$$\mathrm{Codim}_K \bigcap_{i\in I} V_i = m\,,$$
then show that there exists a finite subset $J \subseteq I$ with $|J| \le m$ and $\bigcap_{i\in I} V_i = \bigcap_{i\in J} V_i$. (**Remark :** See also Exercise 4.2. – This statement also hold even if V is not finite dimensional, if we put $\mathrm{Codim}_K U := \mathrm{Dim}_K V/U$, where V/U denote the *quotient space* of V by U.)