E0 219 Linear Algebra and Applications / August-December 2011

(ME, MSc. Ph. D. Programmes)

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212)   **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:30–13:00   **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Jasine Babu** (jasinekb@gmail.com) / **Nitin Singh** (nitin@math.iisc.ernet.in) / **Amulya Ratna Swain** (amulya@csa.iisc.ernet.in) / **Meghana Mande** (meghanamande@gmail.com) / **Achintya Kundu** (achintya.ece@gmail.com)

1-st Midterm : Saturday, September 17, 2011; 10:30 -12:30   **2-nd Midterm :** Saturday, October 22, 2011; 10:30 -12:30
Final Examination : December ??, 2011, 10:00 -13:00

Evaluation Weightage : Assignments : 20%   **Midterms (Two) :** 30%   **Final Examination :** 50%

5. Linear Maps

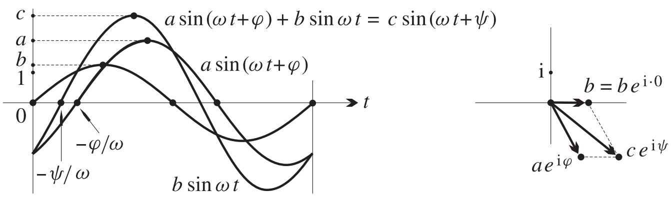
Submit a solution of the $*$-Exercise ONLY

Due Date : Monday, 12-09-2011 (Before the Class)

***5.1** **(a)** (Pointer representation) Let $\omega \in \mathbb{R}_+^\times$ and W be the $\mathbb{R}$-vector space of the functions $a\sin(\omega t + \varphi)$, $a, \varphi \in \mathbb{R}$, with basis $\sin\omega t, \cos\omega t$, (see Exercise 4.1). Then the map

$$\gamma : a\sin(\omega t + \varphi) \longmapsto a e^{\mathrm{i}\varphi},\ a \ge 0,$$

is a $\mathbb{R}$-vector space isomorphism of W onto $\mathbb{C}$. (**Remark :** This isomorphism is called the pointer representation of the simple harmonic motion with the circular frequency ω. The differentiation in W correspond to the multiplication by $\mathrm{i}\omega$ to the pointer representation, i.e. $\gamma(\dot{x}) = \mathrm{i}\omega\,\gamma(x)$ for $x \in W$. In the representation $ae^{\mathrm{i}\varphi}$ of $a\sin(\omega t + \varphi)$, $a \ge 0$, $a = |ae^{\mathrm{i}\varphi}|$ is called the (maximal) amplitude and $e^{\mathrm{i}\varphi}$ is called the phase factor.)



(b) Let $I \subseteq \mathbb{R}$ be an interval with more than one point and let $a \in I$. For $n \in \mathbb{N}^*$, let

$$T_{a,n} : \mathrm{C}^{n-1}_{\mathbb{K}}(I) \to \mathbb{K}[t]_n$$

be the map which maps every function $f \in \mathrm{C}^{n-1}_{\mathbb{K}}(I)$ to its *Taylor-polynomial* of degree $< n$ of f at a, i. e.

$$f \mapsto T_{a,n}(f) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}\,(t-a)^k .$$

Show that $T_{a,n}$ is $\mathbb{K}$-linear. Determine the kernel and the image of this map $T_{a,n}$. (**Remark :** See also Test-Exercise T5.8.)

5.2 Let V be a K-vector space with $\mathrm{Dim}_K V \ge 2$ (i. e. V contain at least two linearly independent vectors). Then every additive map $f : V \to V$ with $f(Kx) \subseteq Kx$ for all $x \in V$ is a homothecy $\vartheta_a : V \to V$, $x \mapsto ax$, of V by a scalar $a \in K$.

5.3 Let $f_1 : V \to V_1$ and $f_2 : V \to V_2$ be homomorphisms of K-vector spaces. The K-linear map $f : V \to V_1 \times V_2$ defined by $f(x) = \big(f_1(x), f_2(x)\big)$ is an isomorphism if and only if f_1 surjective and $f_2|\mathrm{Ker}\, f_1 : \mathrm{Ker}\, f_1 \to V_2$ is bijective.

On the other side one can see auxiliary results and (simple) test-exercises.

Auxiliary Results/Test-Exercises

To understand and appreciate the Test-Exercises which are marked with the symbol † one may possibly require more mathematical maturity than one has! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

Let K be a field.

T5.1 Determine whether the following maps are $\mathbb{R}$-linear:

(a) $f:\mathbb{R}^2 \to \mathbb{R}^2$ with $f(x_1,x_2) := (x_1^2\,,x_2)$. **(b)** $f:\mathbb{R}^2 \to \mathbb{R}^2$ mit $f(x_1,x_2) := (x_1+1\,,0)$.

(c) $f:\mathbb{R}^2 \to \mathbb{R}^2$ with $f(x_1,x_2) := (x_1+x_2\,,x_1)$. **(d)** $f:\mathbb{R}^3 \to \mathbb{R}^2$ with $f(x_1,x_2,x_3) := (|x_1-x_2|\,,2x_3)$.

(e) $f:\mathbb{R}^3 \to \mathbb{R}^2$ with $f(x_1,x_2,x_3) := (3x_1+2x_2\,,x_1+x_3)$.

T5.2 Determine whether the following maps f on the $\mathbb{K}$-vector space $\mathrm{C}^\infty_{\mathbb{K}}(I)$ of infinitely many times differentiable $\mathbb{K}$-valued functions on the interval $I \subseteq \mathbb{R}$ into itself are $\mathbb{K}$-linear:

(a) $f(x) := a_n x^{(n)} + \cdots + a_1\dot{x} + a_0 x + b$ $(a_n,\ldots,a_0,b \in \mathrm{C}^\infty_{\mathbb{K}}(I)$ fixed). **(b)** $f(x) := x^2 + \dot{x}^2$.

(c) $f(x) := \left(t \mapsto x(t_0) + \int_{t_0}^{t} x(\tau)a(\tau)d\tau\right)$ $(t_0 \in I$ and $a \in \mathrm{C}^\infty_{\mathbb{K}}(I)$ fixed).

T5.3 **(a)** The complex conjugation $z \mapsto \bar{z}$ of $\mathbb{C}$ into itself is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

(b) The maps $z \mapsto \operatorname{Re} z$ and $z \mapsto \operatorname{Im} z$ are $\mathbb{R}$-linear forms on $\mathbb{C}$.

T5.4 For the following linear maps f compute the bases for $\operatorname{Ker} f$ and $\operatorname{Im} f$.

(a) $f:\mathbb{R}^3 \to \mathbb{R}^3$ with $f(x_1,x_2,x_3) := (x_1+2x_2+x_3, x_1+3x_2+2x_3, x_1+x_2)$.

(b) $f:\mathbb{R}^4 \to \mathbb{R}^3$ with $f(x_1,x_2,x_3,x_4) := (x_1+3x_2-2x_3+x_4, x_1+4x_2-x_3+3x_4, 2x_1+3x_2-7x_3-4x_4)$.

(c) $f:\mathbb{R}^3 \to \mathbb{R}^4$ with $f(x_1,x_2,x_3) := (x_1+3x_2+3x_3, -2x_1-3x_3, -x_1+x_2-x_3, 3x_1-x_2+4x_3)$.

(d) $f:\mathbb{R}^5 \to \mathbb{R}^4$ with $f(x_1,x_2,x_3,x_4,x_5) :=$
$$(2x_1-x_2-x_3+x_4, -x_1+x_3+x_4+x_5, x_2-x_3-x_4, x_1+x_2-2x_3+x_4+2x_5).$$

T5.5 Let $V := \mathbb{K}[t]$ be the $\mathbb{K}$-vector space of $\mathbb{K}$-valued polynomial functions on $\mathbb{K}$. Which of the following maps $f: V \to V$ are $\mathbb{K}$-linear ? Find the bases for $\operatorname{Ker} f$ and $\operatorname{im} f$ for those f which are $\mathbb{K}$-linear.

(a) $f(x) := x^{(n)} =$ (the n-th derivative of x, $n \in \mathbb{N}$.) **(b)** $f(x) := x(0) + \ddot{x}$. **(c)** $f(x) := \left(t \mapsto \int_0^t \tau\dot{x}(\tau)d\tau\right)$.

(d) $f(x) := P(D)x$, where $P(t) \in \mathbb{K}[t]$ is a monic polynomial[1] and D is the differential operator $x \mapsto \dot{x}$. (**Remark :** See also test-Exercise 3.10.)

T5.6 Let $h: D \to D'$ be an arbitrary map. For every field K, the map $h^*: K^{D'} \to K^D$ defined by $g \mapsto g \circ h$ is K-linear. Describe the functions in $\operatorname{Ker} h^*$ and in $\operatorname{Im} h^*$. Show that h^* is injective (resp. surjective) if and only if h is surjective (resp. injective).

T5.7 A map $f: V \to W$ of $\mathbb{Q}$-vector spaces V and W is already $\mathbb{Q}$-linear if it is additive. The corresponding assertion also holds for vector spaces over the fields $\mathbf{K}_p = \mathbb{Z}/\mathbb{Z}p$, where p is a prime number.

†**T5.8** Let $I \subseteq \mathbb{R}$ be an interval with more than one point and $a \in I$. Let $T_a: \mathrm{C}^\infty_{\mathbb{K}}(I) \to \mathbb{K}[\![t-a]\!]$ be the map which maps every function $f \in \mathrm{C}^\infty_{\mathbb{K}}(I)$ to its Taylor-series of f at a, i.e.
$$T_a(f) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$
Show that T_a is a $\mathbb{K}$-linear map of $\mathrm{C}^\infty_{\mathbb{K}}(I)$ in the space $\mathbb{K}[\![t-a]\!]$ of all (formal) power series in $(t-a)$ with coefficients in $\mathbb{K}$. The kernel of T_a is the space of all plate functions[2] at a. Further, show that T_a

[1] A polynomial $P(t) = \sum_{i=0}^{n} a_i t^i \in K[t]$ of degree n over a field K is called a monic polynomial if the leading co-efficient $a_n = 1$.

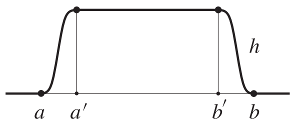
[2] **Plate Functions** Let $f: D \to \mathbb{C}$ be an *analytic*[3] function on an interval $D \subseteq \mathbb{R}$ or a domain $D \subseteq \mathbb{C}$. If the derivatives $f^{(n)}(a)$ of f at a point $a \in D$ are zero, then by the *Taylor's formula* [4] the function f vanishes in a neighbourhood of a and hence by the *identity theorem*[5] f is identically 0 on the whole D. The analogous result does *not* hold for functions

is surjective. (**Remark:** This is precisely the following classical theorem of real analysis which is proved in 1895 by the French mathematician Borel, Émile Félix Édouard-Justin (1871-1956) in his PhD thesis.

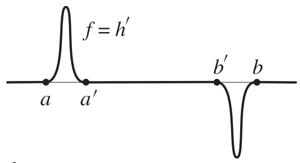
Theorem (Borel) *For every sequence a_n, $n \in \mathbb{N}$, of real or complex numbers there exists an infinitely many times differentiable function f on $\mathbb{R}$ with values in $\mathbb{R}$ resp. $\mathbb{C}$ such that for all $n \in \mathbb{N}$ gilt: $f^{(n)}(0) = a_n$.*

A differentiable function on interval $I \subseteq \mathbb{R}$ can be given by using its derivative f; if f is continuous, then the function ($a \in I$ be a fixed point) $\int_a^x f(t)\,dt$, upto an additive constant, is the required function. This can be generalised, for instance to give a construction of *hat-functions* which are further useful for many constructions in analysis. A function $h : \mathbb{R} \to \mathbb{R}$ is called a hat-function if it satisfies properties stated in the following theorem :

Theorem *Let $a, a', b', b \in \mathbb{R}$ with $a < a' < b' < b$. Then there exists an infinitely many times differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) = 0$ for $t \notin [a,b]$, $h(t) = 1$ for $t \in [a',b']$ and $0 < h(t) < 1$ otherwise.*



Proof The graph of the derivative $f := h'$ of the required function is the following:



Further, we must have $\int_a^{a'} f(t)\,dt = -\int_{b'}^{b} f(t)\,dt = 1$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(t) = 0$ for $t \leq 0$ and $g(t) = e^{-1/t}$ for $t > 0$. Then g is infinitely many times differentiable function. Now, let

$f(t) := \big(g(t-a)g(a'-t)/c\big) - \big(g(t-b')g(b-t)/d\big)$, where $c := \int_a^{a'} g(t-a)\,g(a'-t)\,dt$ and $d := \int_{b'}^{b} g(t-b')\,g(b-t)\,dt$.

Then f is the required function and the function $h(x) := \int_a^x f(t)\,dt$ has the properties stated in the assertion. •

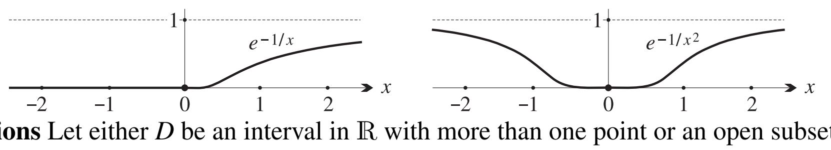
Now using hat-functions, we can give a proof of the Borel's theorem :

---

defined on an interval $I \subseteq \mathbb{R}$, which are infinitely many times differentiable. An infinitely many times differentiable function $f : I \to \mathbb{C}$ is called plate at point $a \in I$, if $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$. There are functions which are plate at a point, but are not identically zero in any neighbourhood of this point. Such a function cannot be analytic; for example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-1/x}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

This function is infinitely many times differentiable and it is plate at 0. It is enough to show that the restricted function $f|\mathbb{R}_+$ is plate at 0. For $x > 0$, we have (can be seen easily by induction on n) $f^{(n)}(x) = h_n(1/x)\exp(-1/x)$ with a monic polynomial function h_n of degree $2n$. Since $\lim_{x \to 0+} h(1/x)\exp(-1/x) = 0$ for *every* polynomial function h, the assertion follows.



[3] **Analytic functions** Let either D be an interval in $\mathbb{R}$ with more than one point or an open subset in in$\mathbb{C}$. A function $f : D \to \mathbb{K}$ is called analytic at a point $a \in D$, if there exists a neighbourhood U of a and a convergent power series $\sum a_k (x-a)^k$ such that $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ for all $x \in U \cap D$. – A function $f : D \to \mathbb{K}$ is called analytic in D, if f is analytic at every point of D.

[4] **Taylor-Formula for analytic functions** Let $f = \sum_{n=0}^{\infty} a_n (x-a)^n$ be the power series expansion of the analytic function $f : D \to \mathbb{C}$ at a point $a \in D$. Then: for every $m \in \mathbb{N}$

$$f^{(m)} = \sum_{n=m}^{\infty} \frac{n}{(n-m)!}\ a_n (x-a)^{n-m}$$

is the power series expansion of the m-th derivative of f at the point $a \in D$. All these power series have the same radius of convergence. In particular, $a_m = \frac{f^{(m)}(a)}{m!}$ for all $m \in \mathbb{N}$.

[5] **Identity theorem for analytic functions** Let D be either an interval in $\mathbb{R}$ or a domain in $\mathbb{C}$. Two analytic functions on D are equal on the whole D if and only if they are equal on a seubset of D, which has at least one limit point in D.

Proof of Borel's theorem : Let $h:\mathbb{R}\to\mathbb{R}$ be an infinitely many times differentiable hat- function with $h(t)=1$ for $|t|\leq 1$ and $h(t)=0$ for all $|t|\geq 2$, further, let $h_n(t):=t^n h(t)$, $n\in\mathbb{N}$. Then $|h_n^{(\nu)}(t)|\leq M_n$ for all $t\in\mathbb{R}$ and all $\nu\in\mathbb{N}$ with $0\leq\nu\leq n$. Put $b_n:=|a_n|M_n+1$ and $f_n(t):=a_n h_n(b_n t)/n!b_n^n$. Then the function $f(t):=\sum_{n=0}^{\infty} f_n(t)$ is a required function. Since $|f_n^{(\nu)}(t)|\leq 1/n!$ for all $n>\nu$ and all $t\in\mathbb{R}$, the series $\sum_{n=0}^{\infty} f_n^{(\nu)}(t)$ of ν-th derivatives is uniformly convergent [6] for every $\nu\in\mathbb{N}$. Therefore by [7] $f^{(\nu)}(t)=\sum_{n=0}^{\infty} f_n^{(\nu)}(t)$ and in particular, $f^{(\nu)}(0)=\sum_{n=0}^{\infty} f_n^{(\nu)}(0)=a_\nu$ for all $\nu\in\mathbb{N}$. •)

T5.9 For every K–vector space V, the map $f\mapsto f(1)$ is a K-isomorphism of $\operatorname{Hom}_K(K,V)$ onto V.

T5.10 Show that the following linear maps $f_i:\mathbb{R}^3\to\mathbb{R}^2$, $i=1,2,3$, in $\operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^3,\mathbb{R}^2)$ are linearly independent: $f_1:(x_1,x_2,x_3)\mapsto(x_1+x_2+x_3,x_1+x_2)$, $f_2:(x_1,x_2,x_3)\mapsto(x_1+x_3,x_1+x_2)$, $f_3:(x_1,x_2,x_3)\mapsto(2x_2,x_1)$.

T5.11 Let K' be a subfield of the field K, V be a K'-vector space and W be a K-vector space. Then W is a K'-vector space in a natural way. With this vector space structure $\operatorname{Hom}_{K'}(V,W)$ is even a K-subspace of W^V.

---

†**T5.12** (Characters) Let M and N be two monoids, then a map $\varphi:M\to N$ is called a (monoid-) homomorphism if $\varphi(xy)=\varphi(x)\varphi(y)$ for all $x,y\in M$ and $\varphi(e_M)=e_N$.

Let M be a monoid and let K be a field. By a character of M in K we mean a homomorphism of M in the multiplicative group $(K,\cdot)$ of K. The map $x\mapsto 1_K$ is a character of M in K, called the trivial character. If $a\in K$, $a\neq$, then the conjugation $\kappa_a=(b\mapsto aba^{-1})$ is a character of the multiplicative monoid of K with values in K.

(a) Let $\varphi_1,\ldots,\varphi_r$ be characters of a monoid M with values in K, i. e. $\varphi_1,\ldots,\varphi_r\in K^M$. Suppose that $\varphi_1,\ldots,\varphi_r$ are linearly independent over K. If a linear combination $\varphi=a_1\varphi_1+\cdots+a_r\varphi_r$ with coefficients $a_1,\ldots,a_r\in K$ is a character of M with values in K, then $\varphi=\kappa_{a_i}\varphi_i$ for every i with $a_i\neq 0$. (**Hint:** Note that: for all $x,y\in M$, one one side we have $\varphi(xy)=a_1\varphi_1(xy)+\cdots+a_r\varphi_r(xy)=a_1\varphi_1(x)\,\varphi_1(y)+\cdots+a_r\varphi_r(x)\,\varphi_r(y)$ and the other-side $\varphi(xy)=\varphi(x)\,\varphi(y)=a_1\varphi(x)\,\varphi_1(y)+\cdots+a_r\varphi(x)\,\varphi_r(y)$.)

(b) (Lemma of Dedekind-Artin[9]) *Let M be a monoid and let K be a field. Then the set of characters of M in K is linearly independent (in the K-vector space K^M of all K-valued functions on M) over K.* (**Hint :** Use part (a) above.)

---

[6] **Uniform convergence** Let D be an arbitrary set and let (f_n) be a sequence of functions $f_n:D\to\mathbb{K}$ on D with values in $\mathbb{K}$.

(1) The sequence (f_n) is called (pointwise) convergent (on D), if there exists a function $f:D\to\mathbb{K}$ with $\lim f_n(x)=f(x)$ for all $x\in D$, i. e. if for every $x\in D$ and for every $\varepsilon>0$ there exists (dependent on x and ε) $n_0\in\mathbb{N}$ such that $|f_n(x)-f(x)|\leq\varepsilon$ for all $n\geq n_0$.

(2) The sequence (f_n) is called uniformly convergent (on D), if there exists a function $f:D\to\mathbb{K}$ such that for every $\varepsilon>0$ there exists (depending only on ε and not on x) $n_0\in\mathbb{N}$ such that $|f_n(x)-f(x)|\leq\varepsilon$ for all $n\geq n_0$.

Uniform convergence of the function sequence (f_n) implies its point-wise convergence. The function f with $f(x)=\lim f_n(x)$ is called the limit function or the limit of the sequence (f_n) and is denoted by $f=\lim_{n\to\infty} f_n=\lim f_n$. For a sequence (f_n) of functions $f_n:D\to\mathbb{K}$, the sequence of partial sums $\sum_{n=0}^{k} f_n$, $k\in\mathbb{N}$, is called the series of the f_n, $n\in\mathbb{N}$. Its limit function (if it exists) it is denoted by $\sum_{n=0}^{\infty} f_n$. If the convergence of partial sums is uniform on D, then we say that the series converges uniformly on D.

[7] **Theorem** *Let D be a domain in $\mathbb{C}$ or an interval in $\mathbb{R}$ and let $f_n:D\to\mathbb{C}$, $n\in\mathbb{N}$, be a sequence of differentiable functions. Further, let $x_0\in D$ be a fixed point. Suppose that:*

(1) *The sequence $f_n(x_0)$, $n\in\mathbb{N}$, is convergent.*

(2) *The sequence f_n', $n\in\mathbb{N}$, of derivatives is locally uniformly convergent [8] on D.*

Then the sequence f_n, $n\in\mathbb{N}$, is locally uniformly convergent on D to a differentiable limit function $f:D\to\mathbb{C}$, and $f'=\lim_{n\to\infty} f_n'$.

[8] **Local Uniform Convergence** A sequence $f_n:D\to\mathbb{K}$, $n\in\mathbb{N}$, of functions on $D\subseteq\mathbb{C}$ is called locally uniform convergent, if for every point $a\in D$ there exists a neighbourhood U of a such that the sequence $f_n|U\cap D$, $n\in\mathbb{N}$, is uniformly convergent on $U\cap D$.

[9] This assertion is used frequently (especially in *Galois Theory*).

(c) Now, let $M = G$ be a group and let K be a field. A character $G \to K$ is then a group homomorphism $G \to K^\times$ and the group $\operatorname{Hom}(G, K^\times)$ of characters is a subgroups of $(K^\times)^G$. *If G is finite and $\chi : G \to K^\times$ is not a trivial character, then* $\sum_{x\in G} \chi(x) = 0$. (**Hint :** If $y \in G$ is an element with $\chi(y) \neq 1_K$, then $\sum_{x\in G} \chi(x) = \sum_{x\in G} \chi(xy) = \left(\sum_{x\in G} \chi(x)\right) \chi(y)$, and hence $\sum_{x\in G} \chi(x) = 0$, since $\chi(y) \neq 1$.)

The group $\operatorname{Hom}(G, \mathbb{C}^\times)$ of characters with values in the field $\mathbb{C}$ is called the character group of G and is denoted by $\hat{G}$. This group plays an important roll is the study of abelian groups.

T5.13 Some simple applications of the Lemma of Dedekind-Artin (see Test-Exercise T5.12-(b)):

(a) Let K be a field. The maps $t \mapsto t^n$, $n \in \mathbb{N}$, are the only polynomial maps of K into itself which are also characters of the multiplicative monoid of K with values in K. More generally: The functions $t \mapsto t^n$, $n \in \mathbb{Z}$, are the only group homomorphisms of $K^\times \to K^\times$, which are also rational functions on $K^\times$. (**Hint :** The case that K is finite should be treated separately; in this case use the fact that the multiplicative group $K^\times$ is cyclic.)

(b) The functions $t \mapsto \exp at$, $a \in \mathbb{C}$, of $\mathbb{R}$ in $\mathbb{C}$ are linearly independent over $\mathbb{C}$.

(c) Let K be a field. The sequences $(a^\nu)_{\nu\in\mathbb{N}}$, $a \in K$, are linearly independent over K. In particular, the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{N}}$ is uncountable dimensional. (**Remark :** See also Exercise 3.4-(b).)

†**T5.14** (Continuous characters of $\mathbb{R}$ and $\mathbb{C}$) **(a)** Every continuous character $\chi : \mathbb{R}^\times \to \mathbb{R}^\times$ is either of the form $x \mapsto |x|^\beta$ or of the form $x \mapsto |x|^\beta \operatorname{Sign} x$ with a (uniquely determined) $\beta \in \mathbb{R}$.

(b) Every continuous character $\chi : \mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z \mapsto |z|^\alpha z^n$ with (uniquely determined) elements $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}$.

(c) The functions $z \mapsto z^n$, $n \in \mathbb{Z}$, are the only continuous endomorphisms of the circle-group $U = \{z \in \mathbb{C} \mid |z| = 1\}$. In particular, identity $(z \mapsto z)$ and the inverse-mapping $(z \mapsto z^{-1})$ are the only continuous automorphisms of U. (**Hint :** Use parts (a) and (b) above.)

†**T5.15** Let U be the circle-group $\{z \in \mathbb{C} \mid |z| = 1\}$.

(a) Every continuous character $U \to \mathbb{C}^\times$ and moreover, every complex-analytic character $\mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z \mapsto z^n$ with a $n \in \mathbb{Z}$.

(b) Every continuous group homomorphism $\mathbb{C}^\times \to U$ is of the form $z \mapsto |z|^{-n+i\gamma} z^n$ with (uniquely determined) $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$.

(c) Every continuous character $(\mathbb{C}, +) \to \mathbb{C}^\times$ is of the form $z \mapsto e^{\alpha z} e^{\beta \bar{z}}$ with (uniquely determined) $\alpha, \beta \in \mathbb{C}$. Further, its image is contained in U respectively, in $\mathbb{R}^\times$, if and only if $\beta = -\bar{\alpha}$ respectively, $\beta = \bar{\alpha}$. Moreover, it is complex-analytic if and only if $\beta = 0$.

(d) Every continuous group homomorphism $\mathbb{C}^\times \to (\mathbb{C}, +)$ is of the form $z \mapsto \beta \ln |z|$ with a $\beta \in \mathbb{C}$. Every continuous group homomorphism $U \to (\mathbb{C}, +)$ and every complex-analytic group homomorphism $\mathbb{C}^\times \to (\mathbb{C}, +)$ is trivial.

T5.16 (Algebras and Algebra homomorphisms) Let K be a field.

(a) (Algebra over K) let A be a K-vector space with a multiplication $A \times A \to A$, is called a K-algebra if the following compatibility conditions hold:

(1) A is a ring with the vector space addition and the given multiplication.
(2) For all $a, b \in K$ and all $x, y \in A$, we have: $(ax)(by) = (ab)(xy)$.

(b) (Algebra-Homomorphisms) If A and B are two K-algebras, then a map $\varphi : A \to B$ is called a K-algebra-Homomorphism if we have:

(1) φ is a K-vector space homomorphism.
(2) φ is compatible with the multiplications on A and B, i. e. $\varphi(xy) = \varphi(x)\,\varphi(y)$ for all $x, y \in A$ and moreover, $\varphi(1_A) = 1_B$.

(c) Every K-algebra homomorphism is, in particular, a ring homomorphism. It follows that:
Let K be a field and let V be a K-vector space. Then $\operatorname{End}_K V$ *is a K-Algebra. The unit group* $(\operatorname{End}_K V)^\times$ *of* $\operatorname{End}_K V$ *is the automorphism-group* $\operatorname{Aut}_K V$ *von V.*

(d) (Function-Algebras) An important class of (commutative) algebras is the class of function-algebras. For an arbitrary field K and an arbitrary set D, the set K^D of all K-valued functions on D is a commutative K-algebra in a natural way and the substitution maps $K^D \to K$, $x \mapsto x(t_0)$, for a fixed $t_0 \in D$, are K-algebra-homomorphisms. All examples of subspaces given in Test-Exercises T2.5 and Exercise 2.1-(a), (b), other than the subspace $K[t]_n$ in Exercise 3.3, are even subalgebras of the algebra of the type K^D. There by a subset A' of a K-algebra A is called a (K-)subalgebra of A, if A' is a K-subspace as well as a subring of A.

T5.17 Let A be a K-Algebra. The map $\lambda : x \mapsto \lambda_x$ (where λ_x is the left-multiplication by x) is an injective K-algebra-homomorphism of A in $\mathrm{End}_K(A)$. Therefore, every K-algebra A is (up to isomorphism) a subalgebra of the endomorphism-algebra of a K-vector space V. Moreover, if A has finite dimension n, then one can also choose V of dimension n.

T5.18 Every 1-dimensional K-algebra is isomorphic to K.

T5.19 Every two-dimensional K-algebra A has a basis of the form $1, x$ and hence it is commutative. The square x^2 is a linear combination $x^2 = \alpha + \beta x$ of 1 and x, and using this equation the multiplication in A is uniquely determined. (Typical Example: $\mathbb{C}$ with the basis $\mathbb{R}$-basis $1, \mathrm{i}$ and the equation $\mathrm{i}^2 = -1$.) The trivial subalgebras $K = K \cdot 1_A$ and A are the only subalgebras of A.

T5.20 Let A be a K-algebra and $x \in A$ be an element. The smallest K-subalgebra of A, containing x, is the subalgebra $K[x] := \sum_{i \in \mathbb{N}} Kx^i$ of all linear combinations of the powers x^i, $i \in \mathbb{N}$, of x. Show that $K[x]$ is a finite dimensional K-algebra if and only if the powers x^i, $i \in \mathbb{N}$, linearly dependent over K.
Moreover, if $K[x]$ is finite dimensional and $\mathrm{Dim}_K K[x] = n$, then $1, x, \ldots, x^{n-1}$ a K-vector space basis of $K[x]$. In this case x is called algebraic over K (of degree n); if $K[x]$ is infinite dimensional, then x is called transcendental over K. If A is finite dimensional with $\mathrm{Dim}_K A = m$, then every element of A is algebraic over K of degree $\leq m$.

T5.21 Let I be a set. Show that:

(a) $K^I = K[x]$ for some $x \in K^I$ if and only if I finite and the map $x : I \to K$ injective.

(b) A map $x \in K^I$ is algebraic over K (see Test-Exercise T5.15) if and only if x attains only finitely many values. Moreover, in this case the degree of x over K is equal to the number of elements $|x(I)|$ of these values.

T5.22 Let I be a finite set and A be a K-subalgebra of the function-algebra K^I. Show that $A = K^I$ if and only if A separates the points of I, i. e. if for every $i, j \in I$ with $i \neq j$, there exists a $x \in A$ such that $x(i) \neq x(j)$. Using this result once more solve the Test-Exercise T5.16-(a). (**Hint :** Suppose that A separates the points. Then, for every fixed $i \in I$, for each $j \neq i$ choose $x_j \in A$ such that $a_j := x_j(j) \neq x_j(i)$. Then $\prod_{j \neq i}(x_j - a_j) \in A$ is a function, which vanishes on $I - \{i\}$ and takes the value $\neq 0$ at i.)

T5.23 Let I be a finite set. For every K-subalgebra A of K^I, let R_A be the equivalence relation on I, defined by $(i, j) \in R_A$ if and only if $f(i) = f(j)$ for every $f \in A$. Conversely, let A_R denote the K-subalgebra of those functions $I \to K$ which are constant on the equivalence classes of R. (The indicator functions e_J of the equivalence classes J form a K-basis of A_R.) Show that the maps $A \mapsto R_A$ and $R \mapsto A_R$ are inverse-maps from the set of all K-subalgebras of K^I onto the set of all equivalence relations on I. In particular, the number of K-subalgebras of K^I is equal to the Bell's number $\beta_{|I|}$. (**Hint :** Apply Test-Exercise T5.17.)

†**T5.24** (Trigonometric Polynomials) Let $\omega \in \mathbb{R}_+^\times$ be fixed. Then the $\mathbb{C}$-subspace $\sum_{n \in \mathbb{Z}} \mathbb{C} e^{i\omega nt}$ is a $\mathbb{C}$-subalgebra of $\mathrm{C}_\mathbb{C}^\omega(\mathbb{R})$. It is the smallest $\mathbb{C}$-subalgebra $\mathbb{C}[\sin \omega t, \cos \omega t]$ of $\mathrm{C}_\mathbb{C}^\omega(\mathbb{R})$, containing the functions $\sin \omega t$ and $\cos \omega t$ and the functions 1; $\sin n\omega t$, $\cos n\omega t$, $n \in \mathbb{N}^*$, form a $\mathbb{C}$-basis. These functions also form a $\mathbb{R}$-basis of the $\mathbb{R}$-subalgebra $\mathbb{R}[\sin \omega t, \cos \omega t]$ of the $\mathbb{R}$-valued functions in $\mathbb{C}[\sin \omega t, \cos \omega t]$. (The algebras $\mathbb{C}[\sin \omega t, \cos \omega t]$ and $\mathbb{R}[\sin \omega t, \cos \omega t]$ are called the algebras of the trigonometric polynomials corresponding to the basic-frequency ω.)

†**T5.25** Let A be a K-algebra and $a \in A^\times$ be a unit in A. Then the map $\kappa_a : x \mapsto axa^{-1}$ from A into itself is an K-algebra-automorphism of A. This is called the conjugation by a or the inner automorphism by a. The map $a \mapsto \kappa_a$ from $A^\times$ into the group of the K-algebra-automorphisms of A is a group homomorphism with the kernel $A^\times \cap \mathrm{Z}(A) = \mathrm{Z}(A)^\times$, where $\mathrm{Z}(A)$ denote the center of A, which is the K-subalgebra of those elements $a \in A$, which commute with *all* elements of A.