E0 219 Linear Algebra and Applications / August-December 2011

(ME, MSc. Ph. D. Programmes)

Download from : `http://www.math.iisc.ernet.in/patil/courses/courses/Current Courses/...`

Tel : +91-(0)80-2293 2239/(Maths Dept. 3212)   **E-mails :** dppatil@csa.iisc.ernet.in / patil@math.iisc.ernet.in

Lectures : Monday and Wednesday ; 11:30–13:00   **Venue:** CSA, Lecture Hall (Room No. 117)

Corrections by : **Jasine Babu** (jasinekb@gmail.com) / **Nitin Singh** (nitin@math.iisc.ernet.in) / **Amulya Ratna Swain** (amulya@csa.iisc.ernet.in) / **Achintya Kundu** (achintya.ece@gmail.com)

1-st Midterm : Saturday, September 17, 2011; 15:00 -17:00   **2-nd Midterm :** Saturday, October 22, 2011; 10:30 -12:30
Final Examination : December ??, 2011, 10:00 -13:00

Evaluation Weightage : Assignments : 20%   **Midterms (Two) :** 30%   **Final Examination :** 50%

7. Direct Sums and Projections ; Dual spaces

Submit a solution of the $*$-Exercise ONLY

Due Date : Monday, 26-09-2011 (Before the Class)

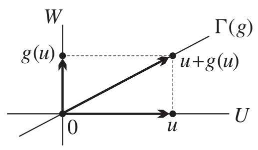
In the following Exercises, let K denote a field and U, V, W denote a K-vector spaces.

7.1 Let $f : U \to V$ and $g : V \to W$ be homomorphisms of K-vector spaces. If gf is an isomorphism of U onto W, then show that V is the direct sum of $\operatorname{Im} f$ and $\operatorname{Ker} g$, i. e. $V = \operatorname{Im} f \oplus \ker g$.

7.2 Assume that K has at least n elements. Let $U_1, \ldots, U_n$ be subspaces (of a finite dimensional K-vector space V) of equal dimension. Then show that $U_1, \ldots, U_n$ have a common complement in V, i. e. $V = U_i \oplus W$ for every $i = 1, \ldots, n$. (**Hint :** Use the Exercise 4.5.)

***7.3** Suppose that the K-vector space V is the direct sum of the subspaces U and W.

(a) For every linear map $g : U \to W$, show that the graph $\Gamma(g) := \{u + g(u) \mid u \in U\} \subseteq V$ of g is a complement of W in V.



(b) Show that the map $\operatorname{Hom}_K(U, W) \to \mathscr{C}(W, V)$ defined by $g \mapsto \Gamma(g)$ is bijective, where $\mathscr{C}(W, V)$ denote the set of all complements of W in V. Describe this bijection for $V = \mathbb{R}^2$ and $U = \mathbb{R} \times \{0\} (=$ x-axis explicitly.

(c) Suppose that $\operatorname{Dim}_K U = \operatorname{Dim}_K W = n$. Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ be bases of U and W, respectively. Then show that $u_1 + w_1, \ldots, u_n + w_n$ is a basis of a complement of U as well as a complement of W in V.

7.4 Let V be a K-vector space and let $f_1, \ldots, f_n \in V^*$ be linear forms on V. Let $f : V \to K^n$ be the homomorphism defined by $f(x) := \big(f_1(x), \ldots, f_n(x)\big)$. Then show that $\operatorname{Dim}(Kf_1 + \cdots + Kf_n) = \operatorname{Dim}(\operatorname{Im} f)$. In particular, $f_1, \ldots, f_n$ are linearly independent if and only if the homomorphism f is surjective.

7.5 A K-linear map $f : V \to W$ be a homomorphism of K-vector spaces is injective (respectively, surjective, bijective) if and only if the dual map $f^* : W^* \to V^*$ is surjective (respectively, injective, bijective) (**Remark:** It is not really necessary to assume that V and W are finite dimensional.)

7.6 Let $x_1, \ldots, x_n$ be all non-zero vectors in a K-vector space V over a field K with $\#K \geq n$. Then Show that there exists a hyperplane H in V such that the vectors $x_i \notin H$ for all $i = 1, \ldots, n$. (**Hint :** There exist a linear form $f_i : V \to K$ such that $f_i(x_i) = 1 \neq 0$ for each $i = 1, \ldots, n$. Therefore the subspaces $(Kx_i)^\circ$, $i = 1, \ldots, n$ are proper subspaces of the K-vector space V^* and hence by Exercise 2.2 $(Kx_1)^\circ \cup \cdots \cup (Kx_n)^\circ \subsetneq V^*$. Then choose $f \in V^* \setminus (Kx_1)^\circ \cup \cdots \cup (Kx_n)^\circ$ and take $H := \operatorname{Ker} f$.)

On the other side one can see auxiliary results and (simple) Test-Exercises.

Auxiliary Results/Test-Exercises

To understand and appreciate the Test-Exercises which are marked with the symbol † one may possibly require more mathematical maturity than one has! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

T7.1 In the following examples determine whether the vector space $\mathbb{R}^3$ respectively $\mathbb{R}^4$ are the direct sums of the subspaces U and W:

(a) $U := \{(a_1,a_2,a_3) \mid a_1+a_2+a_3=0, a_2=a_3\}$ and $W := \{(a_1,a_2,a_3) \mid a_1+2a_2=0, a_1=a_3\}$.

(b) $U := \{(a_1,a_2,a_3) \mid a_1+a_2+a_3=0\}$ and $W := \{(a_1,a_2,a_3) \mid a_1+2a_2=0\}$.

(c) $U := \{(a_1,a_2,a_3) \mid a_1+a_2+a_3=0,\ a_2=a_3\}$, and $W := \{(a_1,a_2,a_3) \mid a_1=a_3\}$.

(d) $U := \{(a_1,a_2,a_3,a_4) \mid a_1+a_3=0,\ a_2+a_4=0\}$, and $W := \{(a_1,a_2,a_3,a_4) \mid a_1+a_2=0,\ a_1+a_4=0\}$.

T7.2 Show that the sum $\sum_{i=1}^n U_i$ of subspaces $U_1,\ldots,U_n$ of the vector space V is direct if and only if $(U_1+\cdots+U_i)\cap U_{i+1}=0$ for $i=1,\ldots,n-1$.

T7.3 Let U_i, $i\in I$ be a family of subspaces of the K-vector space V, let I_j, $j\in J$ be a partition of the indexed set I and let $W_j := \sum_{i\in I_j} U_i$, $j\in J$. The following statements are equivalent:

(i) The sum of the U_i, $i\in I$ is direct.

(ii) For every $j\in J$ the sum of the U_i, $i\in I_j$, is direct and the sum of the W_j, $j\in J$, is direct.

T7.4 Let W be a complement of the subspace U in the vector space V. For every subspace V' of V with $U\subseteq V'$, show that the subspace $W\cap V'$ is a complement of U in V'.

T7.5 Suppose that the vector space V is the direct sum of its subspaces U and W. If $V=U'+W'$ with subspaces $U'\subseteq U$ and $W'\subseteq W$, then show that $U'=U$ and $W'=W$.

T7.6 A linear operator f on a K-vector space V is called an i n v o l u t i o n of V if $f^2=\mathrm{id}_V$. Let $\mathrm{Inv}_K V$ (resp. $\mathrm{Proj}_K V$) denote the set of all involutions (resp. projections) of V. Suppose that $\mathrm{Char}\,K\neq 2$, i.e. $2=1_K+1_K\neq 0$. Then the map $\gamma\colon \mathrm{Proj}_K V\to \mathrm{Inv}_K V$ defined by $p\mapsto \mathrm{id}_V-2p$ is bijective. Further, for $p\in\mathrm{Proj}_K V$ show that

(a) $\mathrm{Im}\,p=\mathrm{Ker}\,(\mathrm{id}+\gamma(p))$ and $\mathrm{Ker}\,p=\mathrm{Ker}\,(\mathrm{id}-\gamma(p))$.

(b) For an involution $f=\gamma(p)$ of V there is a direct sum decomposition :

$$V=V^-\oplus V^+$$

where $V^- := \{x\in V\mid f(x)=-x\}=\mathrm{Im}\,p$ and $V^+ := \{x\in V\mid f(x)=x\}=\mathrm{Ker}\,p$.

T7.7 Suppose that $U_1,\ldots,U_n$ are finite dimensional subspaces of the K-vector space V. Show that

$$\mathrm{Dim}\,(U_1+\cdots+U_n)\leq \mathrm{Dim}\,U_1+\cdots+\mathrm{Dim}\,U_n\,.$$

Moreover, the above inequality is and equality if and only if the sum of the U_i, $i=1,\ldots,n$ is direct.

T7.8 The $\mathbb{K}$-vector space $\mathbb{K}^{\mathbb{R}}$ (resp. $\mathbb{K}^{\mathbb{K}}$) of the $\mathbb{K}$-valued functions on $\mathbb{R}$ (resp. $\mathbb{C}$) is the direct sums of the $\mathbb{K}$-subspaces W_{even} and W_{odd} of all even and all odd functions, respectively. (**Hint :** See Test-Exercise 2.1-d).)

T7.9 Let p be a projection and let f be an arbitrary operator on the K-vector space V.

(a) p and f commute (i.e. $fp=pf$) if and only if the subspaces $\mathrm{Im}\,p$ and $\mathrm{Ker}\,p$ are invariant under f, i. e. $f(\mathrm{Im}\,p)\subseteq\mathrm{Im}\,p$ and $f(\mathrm{Ker}\,p)\subseteq\mathrm{Ker}\,p$.

(b) The subspace $\mathrm{Im}\,p$ is invariant under f if and only if $fp=pfp$.

(c) The subspace $\mathrm{Ker}\,p$ is invariant under f if and only if $pf=pfp$.

T7.10 Let $p_1,\ldots,p_n$ be distinct pairwise commuting projections of the K-vector space V. Then show that the composition $p := p_1\cdots p_n$ is a projection of V with

$$\operatorname{Im} p = (\operatorname{Im} p_1)\cap\cdots\cap(\operatorname{Im} p_n) \quad\text{and}\quad \operatorname{Ker} p = (\operatorname{Ker} p_1)+\cdots+(\operatorname{Ker} p_n)\,.$$

Further, show by examples that the composition p_1p_2 of two projections can be a projection without the condition that p_1 and p_2 commute.

T7.11 Let $p_1,\ldots,p_n$ be distinct pairwise commuting projections of the K-vector space V and let $q_1 := \mathrm{id}_V - p_1,\ldots,q_n := \mathrm{id}_V - p_n$ be the complementary projections.

(a) Show that the projections $p_1,\ldots,p_n$, $q_1,\ldots,q_n$ are pairwise commuting.

(b) For $H=\{i_1,\ldots,i_r\}\subseteq\{1,\ldots,n\}$ with $i_1<\cdots<i_r$, let $p_H := p_{i_1}\cdots p_{i_r}$ and $q_H := q_{i_1}\cdots q_{i_r}$. Show that

$$\mathrm{id}_V = \sum_{H\in\mathfrak{P}(\{1,2,\ldots,n\})} p_H q_{H'}\,, \quad \text{where } H' \text{ denotes the complement } \{1,\ldots,n\}\setminus H \text{ of } H \text{ in} \{1,\ldots,n\}\,.$$

(**Hint :** $\mathrm{id}_V=(p_1+q_1)\cdots(p_n+q_n)$.)

(c) Show that V is the direct sum of the subspaces

$$U_H := \Big(\bigcap_{i\in H}\operatorname{Im} p_i\Big)\cap\Big(\bigcap_{i\notin H}\operatorname{Ker} p_i\Big)\,, H\in\mathfrak{P}(\{1,\ldots,n\})\,.$$

(**Hint :** For $H,L\subseteq\{1,\ldots,n\}$ with $H\neq L$, we have $p_Hq_{H'}p_Lq_{L'}=0$.)

T7.12 Let $p_1,\ldots,p_n$ be distinct pairwise commuting projections of the K-vector space V. Then by Test-Exercise T7.11 (c), V is the direct sums of the subspaces

$$U_1 := \operatorname{Im} p_1\cap\operatorname{Im} p_2\,,\quad U_2 := \operatorname{Im} p_1\cap\operatorname{Ker} p_2\,,$$

$$U_3 := \operatorname{Ker} p_1\cap\operatorname{Im} p_2\,,\quad U_4 := \operatorname{Ker} p_1\cap\operatorname{Ker} p_2\,.$$

For all 16 subsets $S\subseteq\{1,2,3,4\}$ give (with the help of p_1 and p_2) the projection onto $\sum_{i\in S}U_i$ along $\sum_{i\notin S}U_i$.

T7.13 Let p and q be projections of the K-vector space V.

(a) Suppose that $\operatorname{Char} K\neq 2$, i.e. $2=1_K+1_K\neq 0$ in K. Then show that $p+q$ is a projection of V if and only if $pq=qp=0$. Moreover, in this case

$$\operatorname{Im}(p+q)=\operatorname{Im} p\oplus\operatorname{Im} q\,,\quad\text{and}\quad \operatorname{Ker}(p+q)=(\operatorname{Ker} p)\cap(\operatorname{Ker} q)\,.$$

(b) Suppose that $\operatorname{Char} K=2$. Then show that $p+q$ is a projection of V if and only if $pq=qp$. Moreover, in this case

$$\operatorname{Im}(p+q)=(\operatorname{Im} p\cap\operatorname{Ker} q)\oplus(\operatorname{Im} q\cap\operatorname{Ker} p)\ \text{and}\ \operatorname{Ker}(p+q)=(\operatorname{Im} p\cap\operatorname{Im} q)\oplus(\operatorname{Ker} p\cap\operatorname{Ker} q)\,.$$

T7.14 Let p and q be projections of the K-vector space V. Show that p and q have the same image if and only if $pq=q$ and $qp=p$.

T7.15 Suppose that U and U' are two complements of the subspace W of the K-vector space V and p denote the projection of V onto U along W. Then show that $p\upharpoonright U' : U'\to U$ is an isomorphism.

T7.16 Let v_i, $i\in I$ be a basis of the finite dimensional K-vector space V and let U be a subspace of V. Then show that there exists a subset J of I such that the projection p_J onto $V_J := \sum_{i\in J}Kv_i$ along $V_{I\setminus J}=\sum_{i\in I\setminus J}Kv_i$ induces an isomorphism of U onto V_J. (**Remark :** This assertion is true even if I is not a finite set.)

T7.17 Let $f:V\to V'$ be a homomorphism of K-vector spaces. Then show that $W\subseteq V$ is a direct summand of $\operatorname{Ker} f$ in V if and only if f induces an isomorphism $f\upharpoonright W : W\to\operatorname{Im} f$ of W onto $\operatorname{Im} f$.

T7.18 Let V be a K-vector space and let $f_1:U_1\to V$, $f_2:U_2\to V$ be two surjective homomorphisms of K-vector spaces. Further, let $f:U_1\oplus U_2\to V$ be the homomorphism defined by $f(x_1,x_2):=f_1(x_1)+f_2(x_2)$, $x_1\in U_1$, $x_2\in U_2$. Then show that

$$\operatorname{Ker} f_1\oplus U_2\cong\operatorname{Ker} f\cong U_1\oplus\operatorname{Ker} f_2\,.$$

T7.19 Let V be a two dimensional K-vector space with basis x,y. Show that the complements of the line Kx in V are the distinct lines of the form $K(ax+y)$, $a\in K$.

T7.20 Suppose that the K-vector space V is the direct sum of the subspaces U and W. Further, let V' be another K-vector space and let $f:V \to V'$ be a linear map of K-vector spaces such that $f \upharpoonright W: W \to \operatorname{Im} f$ is bijective (see Exercise 7.1). Then show that there exists a unique K-linear map $g:U \to W$ such that $\operatorname{Ker} f = \Gamma(g) = \{u+w \mid u \in U, w = g(u)\}$. (**Remark :** In this case the equation $w = g(u)$ is called the solution of the equation $f(x) = 0$, $x \in V$, along $w \in W$. This is the linear version of the *Theorem on implicit functions* from Analysis.)

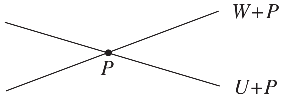
T7.21 Let V be a finite dimensional K-vector space and let $f : V \to V$ be an operator on V. Show that f is a projection of V if and only if there exists a basis $x_1, \ldots, x_n$ of V such that $f(x_i) = x_i$, $i = 1, \ldots, r$, and $f(x_i) = 0$, $i = r+1, \ldots, n$. (**Remark :** Analogous assertion holds even if V is not finite dimensional, formulate this assertion and prove it.)

T7.22 Let V be a finite dimensional K-vector space and let $f : V \to V$ be an arbitrary operator on V. Show that there exists an automorphism $g : V \to V$ of V and projections $p, q : V \to V$ on V such that $f = pg = gq$. (**Hint :** Extend a basis of Kern f to a basis of V. – In general, such a representation does not exists for operators on infinite dimensional vector spaces. Example?)

†**T7.23** Let E be an affine space over the K-vector space V and let U, W be subspaces of V. Show that

(a) Any two affine subspaces F and F' of E which are parallel to U and W, respectively, intersects if and only if V is the sum of U and W.

(b) Any two affine subspaces F and F' of E which are parallel to U and W, respectively, intersects exactly in a point if and only if V is the direct sum of U and W.



T7.24 Let $f: V \to V''$ be a surjective K-linear map, let $U \subseteq V$ be a K-subspace of V and let $f|U: U \to V''$ be the restriction of f to U. Then show that

(a) $f \upharpoonright U$ is injective if and only if $U \cap \operatorname{Ker} f = 0$.
(b) $f \upharpoonright U$ is surjective if and only if $U + \operatorname{Ker} f = V$.

(c) $f \upharpoonright U$ is an isomorphism if and only if $V = U \oplus \operatorname{Ker} f$, i.e. U is a complement of $\operatorname{Ker} f$ in V.

†**T7.25** Let $f: V \to V''$ be a surjective K-linear map and let W be its kernel. Then the set of all complements U of W in V is an affine space over the K-vector space $\operatorname{Hom}_K(V'', W)$ with respect to the operation $\operatorname{Hom}_K(V'', W) \times \mathscr{C}(W, V) \to \mathscr{C}(W, V)$, $(h, U) \longmapsto h + U := \{h(f(x)) + x \mid x \in U\}$, $h \in \operatorname{Hom}_K(V'', W)$.

T7.26 For a subspace U of V, the following statements are equivalent:
(i) $U \neq V$ and there exists a $v \in V$ such that $V = U + Kv$.
(i') There exists a $v \in V$, $v \neq 0$ such that $V = U \oplus Kv$.
(ii) There exists a linear form $f \neq 0$ on V such that $U = \operatorname{Kern} f$. (**Remark:** The subspaces U with these properties are called hyperplanes in V.)

T7.27 Suppose that V is *not* finite dimensional and let v_i, $i \in I$ be a basis of V. Further, let v_i^*, $i \in I$ be the coordinate functions with respect to the basis v_i $i \in I$ and $W := \sum_{i \in I} K v_i^* \subseteq V^*$ be the subspace of V^* generated by v_i^*, $i \in I$. (Consider in particular, the concrete situation $V := K^{(I)}$, $v_i := e_i$, $i \in I$ with $V^* \cong K^I$, $W \cong K^{(I)} \subset K^I$.)

(a) The linear form $\sum_{i \in I} a_i v_i \longmapsto \sum_{i \in I} a_i$ on V does not belong to W. In particular, $W \neq V^*$ and v_i^*, $i \in I$ *not* basis of V^*.

(b) ${}^{\circ}W = 0$ and so $({}^{\circ}W)^{\circ} = V^* \neq W$.

(c) The canonical homomorphism $\sigma_V : V \to V^{**}$ is not surjective.

T7.28 Let $v_1, \ldots, v_n$ be a basis of V. For $a_1, \ldots, a_n \in K$, find a basis of the kernel of the linear form $a_1 v_1^* + \cdots + a_n v_n^*$.

T7.29 If V^* is finite dimensional, then V is finite dimensional.

T7.30 Suppose that V is a finite dimensional. Then show that for every basis f_i, $i \in I$ of V^*, there exists a (unique) basis v_i, $i \in I$ of V such that $f_i = v_i^*$, $i \in I$.

T7.31 Suppose that V is a finite dimensional. Then show that $\operatorname{Dim} U = \operatorname{Codim}(U^\circ, V^*)$ for every subspace $U \subseteq V$. (**Remark :** It is enough to assume that U is finite dimensional.)

T7.32 Suppose that V is a finite dimensional. For subspaces $U_1, U_2 \subseteq V$ (respectively, $W_1, W_2 \subseteq V^*$), show that (i) $(U_1+U_2)^\circ = U_1^\circ \cap U_2^\circ$, (ii) $(U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ$, (iii) ${}^\circ(W_1+W_2) = {}^\circ W_1 \cap {}^\circ W_2$, (iv) ${}^\circ(W_1 \cap W_2) = {}^\circ W_1 + {}^\circ W_2$.

T7.33 Let $r \in \mathbb{N}$. The maps $W \mapsto {}^\circ W$ and $U \mapsto U^\circ$ are inverses of each other on the set of all r-dimensional subspaces W of V^* and the set of all r-codimensional subspaces U of V. (**Remark:** A subspace $U \subseteq V$ is called r-codimensional in V if one (and hence every) of the complement of U in V is r-dimensional. – the map $U \mapsto U^\circ$ from the set of all r-dimensional subspace U of V into the set of all r-codimensional subspaces of V^* is injective by 5.G.7, see Test-Exercise T7.31. But not surjective in the case when V is not finite dimensional.)

T7.34 A K-linear map $f : V \to W$ be a homomorphism of K-vector spaces is equal to 0 if and only if the dual map $f^* : W^* \to V^*$ is the 0 map.

T7.35 Let $f : V \to W$ be a homomorphism of K-vector spaces. The kernel of the dual map $f^* : W^* \to V^*$ is the space of all linear forms $g : W \to K$ on W, which vanish on the $\operatorname{Im} f$ and so $\operatorname{Ker} f^* = (\operatorname{Im} f)^\circ$. The image of f^* is the space of all linear forms $V \to K$, which vanish on the $\operatorname{Ker} f$ and so $\operatorname{Im} f^* = (\operatorname{Ker} f)^\circ$.

T7.36 Let K be a subfield of the field L.

(a) A family $f_i \in K^D$, $i \in I$ of K-valued functions on D is linearly independent over K if and only if the family f_i, $i \in I$ as a family of L-valued functions on D is linearly independent over L. Further, show that

$$\operatorname{Dim}_K\big(\textstyle\sum_{i\in I} K f_i\big) = \operatorname{Dim}_L\big(\textstyle\sum_{i\in I} L f_i\big) \text{ for an arbitrary family } f_i \in K^D,\ i \in I.$$

(b) Let W be a K-subspace of the K-vector space K^D and $L \cdot W$ be the L-subspace of the L-vector space L^D generated by W. Then show that $K^D \cap L \cdot W = W$. (**Hint :** Let $f \in K^D \cap LW$, but $f \notin W$. Then f can be expressed as $f = c_1 f_1 + \cdots + c_r f_r$ with $c_1, \ldots, c_r \in L$ and linear independent functions $f_1, \ldots, f_r \in W$. Then $f, f_1, \ldots, f_r$ are linearly independent over K, but are linearly dependent over L, a contradiction!)

T7.37 ($\mathbb{C}$-anti-linear forms) Let V be a $\mathbb{C}$-Vector space. A $\mathbb{C}$-anti-linear map $V \to \mathbb{C}$ is called a $\mathbb{C}$-anti-linear form on V. The $\mathbb{C}$-vector space of the $\mathbb{C}$-anti-linear forms on V is denoted by $\overline{V}^*$.

(a) $f : V \to \mathbb{C}$ is linear over $\mathbb{C}$ if and only if $\overline{f} : V \to \mathbb{C}$ ($x \mapsto \overline{f(x)}$) is $\mathbb{C}$-anti-linear. The linear forms $f_i \in V^*$, $i \in I$ form a $\mathbb{C}$-basis of V^* if and only if the $\mathbb{C}$-anti-linear forms $\overline{f_i}$, $i \in I$ form a $\mathbb{C}$-basis of $\overline{V}^*$.

(b) If v_i, $i \in I$ is a finite $\mathbb{C}$-basis of V, then $\overline{v_i^*}$, $i \in I$ is a $\mathbb{C}$-basis of $\overline{V}^*$. In particular, $\operatorname{Dim}_{\mathbb{C}} V = \operatorname{Dim}_{\mathbb{C}} V^* = \operatorname{Dim}_{\mathbb{C}} \overline{V}^*$ for every finite dimensional $\mathbb{C}$-vector spaces V.

(c) $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^* \oplus \overline{V}^*$ $(\subseteq \mathbb{C}^V)$.

†**T7.38** Let $K \subseteq L$ be a field extension and let V be a L-vector space (and hence it is also a K-vector space by the restriction of scalars). Further, let $\sigma : L \to K$ be a K-linear form $\neq 0$. (**Remark:** Such a function is also called a generalised trace function. In the case $\mathbb{R} \subseteq \mathbb{C}$ one may choose $\sigma := \operatorname{Re}$. The meaning of trace in this case is $2\operatorname{Re}$, see Exercise ???) $\operatorname{Hom}_K(V, K)$ is L-vector space with scalar multiplication $(bf)(x) := f(bx)$ for $b \in L$, $x \in V$ and $f \in \operatorname{Hom}_K(V, K)$.

(a) Let $[L : K] < \infty$. Then the map $\operatorname{Hom}_L(V, L) \xrightarrow{\sim} \operatorname{Hom}_K(V, K)$ defined by $f \mapsto \sigma \circ f$ is an isomorphism of L-vector spaces. (**Hint :** With the help of a L-basis of V one can reduce to the case $V = L$. In this case use a dimension-argument. In the case $\mathbb{R} \subseteq \mathbb{C}$ and $\sigma := \operatorname{Re}$ the map $g \longmapsto \big(x \mapsto g(x) - \mathrm{i}\, g(\mathrm{i}x)\big)$ is the inverse map.)

(b) If $[L : K] < \infty$. Then every K-subspace $U \subseteq V$ with $\operatorname{Codim}_K(U, V) = r \in \mathbb{N}$ is contain a L-subspace U' with $\operatorname{Codim}_L(U', V) \le r$. (See Test-Exercise T7.33.)

(c) There exists a $\mathbb{Q}$-hyperplane H in $\mathbb{R}^2$ such that H do not contain any $\mathbb{R}$-hyperplane in $\mathbb{R}^2$.

†**T7.39** Let K be a finite field with $\operatorname{card}(K) = q$ (note that $q = p^m$ for some $m \in \mathbb{N}^+$, where $p := \operatorname{Char} K$) and let V be an n-dimensional K-vector space.

(a) For $n \in \mathbb{N}$, let $\alpha_q(n,r)$ be the number of linearly independent r-tuples $(x_1,\ldots,x_r) \in V^r$. For $1 \le r \le n$, show that

$$\alpha_q(n,r) = q^{(r-1)r/2} \prod_{i=n-r+1}^{n} (q^i - 1)\,.$$

In particular, $\alpha_q(n,r)$ depends only on q,n,r and does not depend on K and V. (**Hint :** Use induction on r.)

(b) $\operatorname{card}(\operatorname{End}_K(V)) = q^{n^2}$ and $\operatorname{card}(\operatorname{Aut}_K(V)) = \alpha_q(n,n)$.

(c) For $n \in \mathbb{N}$, let $\beta_q(n,r)$ be the number of r-dimensional K-subspaces of V. For $1 \le r \le n$, show that $\operatorname{Char} K$ does not divide $\beta_q(n,r)$ and $\beta_q(n,r) = \alpha_q(n,r)\alpha_q(r,r)^{-1}$. In particular, $\beta_q(n,r)$ depends only on q,n,r and does not depend on K and V.

(d) The number of projections of V are $\sum_{r=0}^{n} \beta_q(n,r) q^{r(n-r)}$.

(e) Let H be an *elementary abelian p–group* [1] of order p^n, where p is a prime number. Compute the number of endomorphisms and automorphisms of H and the number of subgroups.

(f) Let p be a prime number and let $n \in \mathbb{N}$. For $r \in \mathbb{Z}$, let $\begin{bmatrix} n \\ r \end{bmatrix}$ denote the number of subgroups of order p^r in an elementary abelian p–group of order p^n. This number is 0 for $r < 0$ and $r > n$; further,

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(p^n-1)(p^{n-1}-1)\cdots(p^{n-r+1}-1)}{(p-1)(p^2-1)\cdots(p^r-1)}$$

for $0 \le r \le n$. (**Remark :** One can define these numbers by the above properties without any reference to the groups – and vector spaces. Note the similarity between these numbers and the binomial coefficients : $\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}$, and for $n \ge 1$, we have the recursion formula : $\begin{bmatrix} n \\ r \end{bmatrix} = p^r \begin{bmatrix} n-1 \\ r \end{bmatrix} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}$.)

(g) In the set of subspaces of V which is ordered by the inclusion, the maximal number of elements which are not comparable is $\beta_q(n,[n/2])$.

[1] The additive groups or the vector spaces over the field $\mathbf{K}_p = \mathbb{Z}/\mathbb{Z}p$ are called the e l e m e n t a r y a b e l i a n p - g r o u p s .