E0 219 Linear Algebra and Applications / August-December 2011

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00   **Venue:** CSA, Lecture Hall (Room No. 117)

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1-st Midterm : Saturday, September 17, 2011; 15:00 -17:00   **2-nd Midterm :** Saturday, October 22, 2011; 10:30 -12:30

Final Examination : December ??, 2011, 10:00 -13:00

Evaluation Weightage : Assignments : 20%   **Midterms (Two) :** 30%   **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

8. Matrices

Submit a solution of the $*$-Exercise ONLY

Due Date : Monday, 10-10-2011 (Before the Class)

Complete correct solution of the $**$-Exercise (Exercise 8.8) carries 10 Bonus Points!

8.1 Let V be a vector space of dimension n over a field K and let $f \in \operatorname{End}_K V$. Then there exists a basis of V such that the matrix $\mathfrak{M}_{\mathfrak{v}}^{\mathfrak{v}}(f)$ of f with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix},$$

where the elements $a_{21}, a_{32}, \ldots, a_{n,n-1}$ below the main-diagonal are either 1 or 0. (**Remark :** A matrix of this form, where the elements $a_{21}, a_{32}, \ldots, a_{n,n-1}$ are arbitrary is called a Hessenberg[1]–matrix. The existence of such a matrix representation is much simpler than what the applied mathematicians will make you think, when they are using *Householder type reflections*[2] for the construction (which works over $\mathbb{C}$ only), see also[3]. However, it is much simpler to construct a basis $w_1, \ldots, w_n$ of V (over arbitrary field K), see the – **Proof:** To construct a basis $w_1, \ldots, w_n$ of V (over arbitrary field K), choose any $w_1 \neq 0$ in V. If $f(w_1) \in Kw_1$, say $f(w_1) = a_1 w_1$, then choose $w_2 \in V$, $w_2 \notin Kw_1$, and take $a_{11} := a_1, a_{i,1} := 0$ for $i = 2, \ldots, n$. If $f(w_1) \notin Kw_1$, put $w_2 := f(w_1)$ and $a_{11} := 0, a_{21} := 1$, $a_{i,1} = 0$ for $i = 3, \ldots, n$. Then w_1, w_2 are linearly independent, and the first column of the matrix will have the required form. Now, assume that we have chosen linearly independent vectors $w_1, \ldots, w_j$, $j < n$, such that the first $j-1$ columns of the matrix have the right form. Then proceed as follows: If $f(w_j) \in Kw_1 + \cdots Kw_j$, say $f(w_j) = a_1 w_1 + \cdots + a_j w_j$, choose a vector $w_{j+1} \in V$, $w_{j+1} \notin Kw_1 + \cdots + Kw_j$, and put $a_{ij} := a_i$ for $i = 1, \ldots, j$ and $a_{ij} := 0$ for $i = j+1, \ldots, n$. If $f(w_j) \notin Kw_1 + \cdots Kw_j$, put $w_{j+1} := f(w_j)$ and $a_{ij} := 0$ for $i = 1, \ldots j$, $a_{j+1,j} := 1$ and

[1] Hessenberg matrices were first investigated by Karl Hessenberg (1904-1959), a German engineer whose dissertation investigated the computation of eigenvalues and eigenvectors of linear operators, see [Hessenberg, K. Thesis. Darmstadt, Germany: Technische Hochschule, 1942.]

[2] Householder transformation was introduced in 1958 by Alston Scott Householder (1904-1993) an American mathematician who specialized in mathematical biology and numerical analysis.

[3] [Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Reduction of a General Matrix to Hessenberg Form." § 11.5 in Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed. Cambridge, England: Cambridge University Press, pp. 476-480, 1992.]

$a_{ij} = 0$ for $i = j+2,\ldots,n$.. Then $w_1,\ldots,w_{j+1}$ are linearly independent, and the first j columns of the matrix will have the required form. This method stops after having chosen w_n, because there are no requirements on the last column of that matrix. •)

*__8.2__ **(a)** (Binomial inversion formula) Let $n\in\mathbb{N}$. From the equations

$$(1+t)^j = \sum_{i=0}^{j}\binom{j}{i}t^i,\quad t^j = (1+t-1)^j = \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}(1+t)^i,\qquad j=0,\ldots,n,$$

deduce that the matrices

$$\begin{pmatrix}\binom{0}{0} & \binom{1}{0} & \cdots & \binom{n}{0}\\ 0 & \binom{1}{1} & \cdots & \binom{n}{1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \binom{n}{n}\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\binom{0}{0} & -\binom{1}{0} & \cdots & (-1)^n\binom{n}{0}\\ 0 & \binom{1}{1} & \cdots & (-1)^{n-1}\binom{n}{1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \binom{n}{n}\end{pmatrix}$$

in $\mathrm{M}_{n+1}(K)$ are inverses of each other.

(b) (Fourier-inversion formula) Let $n\in\mathbb{N}^*$ and ζ be a primitive n-th root of unity, for example, $\zeta := \exp(2\pi\mathrm{i}/n)$. Then the matrices

$$(\zeta^{\mu\nu})_{0\le\mu,\nu<n}\quad\text{and}\quad\frac{1}{n}(\zeta^{-\mu\nu})_{0\le\mu,\nu<n}$$

are inverses of each other in $\mathrm{M}_n(\mathbb{C})$. (**Proof:** We have to show that $\sum_{\nu=0}^{n-1}\zeta^{\mu\nu}\,\frac{1}{n}\,\zeta^{-\nu\lambda} = \delta_{\mu\lambda}$. For $\lambda=\mu$ indeed $\sum_{\nu=0}^{n-1}\zeta^{\mu\nu}\,\frac{1}{n}\,\zeta^{-\nu\mu} = \sum_{\nu=0}^{n-1}\frac{1}{n} = 1$. For $\lambda\neq\mu$ we have $\sum_{\nu=0}^{n-1}\zeta^{\mu\nu}\,\frac{1}{n}\,\zeta^{-\nu\lambda} = \frac{1}{n}\sum_{\nu=0}^{n-1}\left(\zeta^{\mu-\lambda}\right)^{\nu} = \frac{1}{n}\,\frac{1-\left(\zeta^{\mu-\lambda}\right)^n}{1-\zeta^{\mu-\lambda}} = \frac{1-(\zeta^n)^{\mu-\lambda}}{n(1-\zeta^{\mu-\lambda})} = \frac{1-1^{\mu-\lambda}}{n(1-\zeta^{\mu-\lambda})} = 0$. – **Remark:** More generally, the same assertion holds for an arbitrary field K. – We say that an element $\zeta\in K$ is a primitive n-th root of unity if ζ generates a subgroup of order n in the multiplicative group $K^\times$ of the field K, for example, $\zeta := \exp(2\pi\mathrm{i}/n)\in\mathbb{C}$ is a primitive root of unity in the field $\mathbb{C}$. Note that $n\neq 0$ in K. Otherwise K will have a prime characteristic $p=\mathrm{Char}\,K$ which is a divisor of n, i. e. $n = pm$ with $m\in\mathbb{N}$ and $(\zeta^m-1)^p = \zeta^{mp}-1 = \zeta^n-1 = 0$ and hence $\zeta^m-1=0$ a contradiction to the hypothesis that ζ is a primitive n-th root of unity.)

8.3 (Vandermonde-matrices[4]) Let $\lambda_0,\ldots,\lambda_n$ be pairwise distinct elements of the field K. For $j=0,\ldots,n$, let $f_j(t) = \prod_{i\neq j}\frac{(t-\lambda_i)}{(\lambda_j-\lambda_i)} = a_{0j}+a_{1j}t+\cdots+a_{nj}t^n$. Then the matrices

$$(\lambda_i^j) = \begin{pmatrix}1 & \lambda_0 & \cdots & \lambda_0^n\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_n & \cdots & \lambda_n^n\end{pmatrix}\quad\text{and}\quad(a_{ij}) = \begin{pmatrix}a_{00} & \cdots & a_{0n}\\ \vdots & \ddots & \vdots\\ a_{n0} & \cdots & a_{nn}\end{pmatrix}$$

in $\mathrm{M}_{n+1}(K)$ are inverses of each other. (**Hint :** Both these matrices are the transition matrices from the basis $\mathfrak{t} := \{1,t,\ldots,t^n\}$ to the basis $\mathfrak{f} := \{f_0,\ldots,f_n\}$ (check this!) of the space $V = K^{\{\lambda_0,\ldots,\lambda_n\}}$ of K-valued functions on the set $\{\lambda_0,\ldots,\lambda_n\}$ and the other way, respectively, i. e. $\mathfrak{M}_{\mathfrak{t}}^{\mathfrak{f}}(\mathrm{id}_V) = (a_{ij})$ and $\mathfrak{M}_{\mathfrak{f}}^{\mathfrak{t}}(\mathrm{id}_V) = \left(\lambda_i^j\right)$ – Matrices of this type (λ_i^j) are called Vandermonde's matrices.)

[4] In linear algebra, a Vandermonde matrix, named after Alexandre-Théophile Vandermonde (1735-1796), who was a French musician, mathematician and chemist who worked with Bézout and Lavoisier; his name is now principally associated with determinant theory in mathematics. Vandermonde was a violinist, and became engaged with mathematics only around 1770.

8.4 (Cauchy-matrices[5]) Let $\lambda_1,\ldots,\lambda_n$ resp. $\mu_1,\ldots,\mu_n$ be pairwise distinct elements of the field K such that $\lambda_i+\mu_j\neq 0$ for all $i,j=1,\ldots,n$. Let $g(t):=(t+\mu_1)\cdots(t+\mu_n)$ and

$$f_j(t)=\frac{g(\lambda_j)\prod_{i\neq j}(t-\lambda_i)}{g(t)\prod_{i\neq j}(\lambda_j-\lambda_i)}=\sum_{i=1}^{n}\frac{a_{ij}}{t+\mu_i}\qquad\text{(partial fraction decompostion)}.$$

Then the matrices

$$\left(\frac{1}{\lambda_i+\mu_j}\right)=\begin{pmatrix}\frac{1}{\lambda_1+\mu_1} & \cdots & \frac{1}{\lambda_1+\mu_n}\\ \vdots & \ddots & \vdots\\ \frac{1}{\lambda_n+\mu_1} & \cdots & \frac{1}{\lambda_n+\mu_n}\end{pmatrix}\quad\text{and}\quad (a_{ij})=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$$

in $\mathrm{M}_n(K)$ are inverses of each other. Compute the elements a_{ij} explicitly. (**Hint :** For the calculation of the coefficients a_{ij}, we shall use the method of calculation of the coefficient of $1/(t+\mu_i)$ in the partial fraction decomposition of $f_j(t)$ and rewrite the result by using the substitutions of the polynomial $h(t):=(t+\lambda_1)\cdots(t+\lambda_n)$:

$$a_{ij}=\frac{g(\lambda_j)\prod_{\ell\neq j}(-\mu_i-\lambda_\ell)}{g'(-\mu_i)\prod_{\ell\neq j}(\lambda_j-\lambda_\ell)}=\frac{1}{(\mu_i+\lambda_j)}\,\frac{g(\lambda_j)}{g'(-\mu_i)}\,\frac{(-1)^{n-1}h(\mu_i)}{(-1)^{n-1}h'(-\lambda_j)}=\frac{1}{(\mu_i+\lambda_j)}\,\frac{g(\lambda_j)}{g'(-\mu_i)}\,\frac{h(\mu_i)}{h'(-\lambda_j)}\,.$$

Now, by the choice of the a_{ij}, the (k,j)-th coefficient of the matrix-product $\big(1/(\lambda_k+\mu_i)\big)\,(a_{ij})$ is

$$\sum_{i=1}^{n}\frac{1}{\lambda_k+\mu_i}\cdot a_{ij}=f_j(\lambda_k)=\frac{g(\lambda_j)\prod_{i\neq j}(\lambda_k-\lambda_i)}{g(\lambda_k)\prod_{i\neq j}(\lambda_j-\lambda_i)}=\delta_{kj}\,,$$

since numerator and denominator of the fractions are equal for $k=j$ and if $k\neq j$ the product in the numerator is zero. – Matrices of the type $\left(\frac{1}{\lambda_i+\mu_j}\right)_{\substack{1\le i\le m,\\ 1\le j\le n}}$, with with distinct elements $\lambda_1,\ldots,\lambda_m\in K$ and distinct elements $\mu_1,\ldots,\mu_n\in K$, are called Cauchy-matrices. The Hilbert-matrix is a special case of the Cauchy matrix, where $\lambda_i+\mu_j=i+j-1$. Every submatrix of a Cauchy matrix is itself a Cauchy matrix.)

*__8.5__ Compute the inverse of the matrix (called the Heisenberg-matrix[6]) of the form

$$\mathfrak{B}=\begin{pmatrix}1 & a_1 & a_2 & \cdots & a_n & c\\ 0 & 1 & 0 & \cdots & 0 & b_1\\ 0 & 0 & 1 & \cdots & 0 & b_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & b_n\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}\in\mathrm{M}_{n+2}(K)\,.$$

(**Hint:** Let $w_0,\ldots,w_{n+1}$ be a basis of the $n+2$-dimensional vector space V over K. Then $v_0:=w_0$, $v_j:=w_j+a_iw_0$, $j=1,\ldots,n$ and $v_{n+1}:=w_{n+1}+b_nw_n+\cdots+b_1w_1+cw_0$ is also a basis (see Exercise 3.2) of V over K. Further, the Heisenberg-matrix $\mathfrak{B}=\mathfrak{M}^{\mathfrak{w}}_{\mathfrak{v}}$ is the transition matrix of the basis $v_0,\ldots,v_{n+1}$ onto the

[5]Named after Baron Augustin-Louis Cauchy (1789-1857) a French mathematician who was an early pioneer of analysis. He started the project of formulating and proving the theorems of infinitesimal calculus in a rigorous manner, rejecting the heuristic principle of the generality of algebra exploited by earlier authors. He defined continuity in terms of infinitesimals and gave several important theorems in complex analysis and initiated the study of permutation groups in abstract algebra. A profound mathematician, Cauchy exercised a great influence over his contemporaries and successors. His writings cover the entire range of mathematics and mathematical physics.

[6]These matrices were first investigated by Werner Heisenberg (1901-1976) a a German theoretical physicist who made foundational contributions to quantum mechanics and is best known for asserting the uncertainty principle of quantum theory. Matrix mechanics is a formulation of quantum mechanics created by Werner Heisenberg, Max Born, and Pascual Jordan in 1925. Matrix mechanics was the first complete and correct definition of quantum mechanics. It extended the Bohr Model by interpreting the physical properties of particles as matrices that evolve in time. It is equivalent to the Schrödinger wave formulation of quantum mechanics, and is the basis of Dirac's bra-ket notation for the wave function.

basis $w_0,\ldots,w_{n+1}$. Therefore the inverse $\mathfrak{B}^{-1}$ is the transition matrix of the basis $w_0,\ldots,w_{n+1}$ onto the basis $v_0,\ldots,v_{n+1}$. Then the inverse $\mathfrak{B}^{-1}=\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{w}}$ is the transition matrix of the basis $v_0,\ldots,v_{n+1}$ onto the basis $w_0,\ldots,w_{n+1}$. Since $w_0=v_0$, $w_j=v_j-a_iv_0$, $j=1,\ldots,n$ and $w_{n+1}=v_{n+1}-(b_nv_n-a_nv_0)-\cdots-b_1(v_1-a_1v_0)-cv_0=v_{n+1}-b_nv_n-\cdots-b_1v_1+(b_na_n+\cdots+b_1a_1-c)v_0$, it follows that

$$\mathfrak{B}^{-1}=\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{w}}=\begin{pmatrix}1 & -a_1 & -a_2 & \cdots & -a_n & b_na_n+\cdots+b_1a_1-c\\ 0 & 1 & 0 & \cdots & 0 & -b_1\\ 0 & 0 & 1 & \cdots & 0 & -b_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -b_n\\ 0 & 0 & 0 & \cdots & 0 & 1\end{pmatrix}\in \mathrm{M}_{n+2}(K)\,.)$$

8.6 Let I,J be finite sets. Two matrices $\mathfrak{A},\mathfrak{A}'\in\mathrm{M}_{I,J}(K)$ have the same rank if and only if there exist invertible matrices $\mathfrak{B}\in\mathrm{GL}_I(K)$ and $\mathfrak{C}\in\mathrm{GL}_J(K)$ such that $\mathfrak{A}'=\mathfrak{B}\mathfrak{A}\mathfrak{C}$.
(**Hint:** Let $f,g\colon K^J\to K^I$ be linear maps defined by $f(\mathfrak{x}):=\mathfrak{A}\mathfrak{x}$ and $f'(\mathfrak{x}):=\mathfrak{A}'\mathfrak{x}$, $\mathfrak{x}$ is column-vector in K^J, and let $\mathfrak{A}$ respectively $\mathfrak{A}'$ be the matrices with respect to the standard bases. Let $\operatorname{Rank}\mathfrak{A}=\operatorname{Rank}\mathfrak{A}'=r$, and so $\operatorname{Rank}f=\operatorname{Rank}f'=r$. By the proof of the Rank-Theoremthere exist a basis $v_1,\ldots,v_n$ of K^J and a basis $v'_1,\ldots,v'_n$ of K^J such that $w_1:=f(v_1),\ldots,w_r:=f(v_r)$ is a basis of $\operatorname{Im}f$ and $w'_1:=f(v'_1),\ldots,w'_r:=f(v'_r)$ is a basis of $\operatorname{Im}f'$ and that $v_{r+1},\ldots,v_n$ and $v'_{r+1},\ldots,v'_n$ are bases of $\operatorname{Ker}f$ respectively $\operatorname{Ker}f'$. We also extend $w_1,\ldots,w_r$ and $w'_1,\ldots,w'_r$ to bases $w_1,\ldots,w_m$ respectively $w'_1,\ldots,w'_m$ of K^I. Now, we define isomorphisms $h\colon K^J\to K^J$ and $g\colon K^I\to K^I$ by $h(v_i):=v'_i$, $i=1,\ldots,n$, and $g(w_i):=w'_i$, $i=1,\ldots,m$. By construction, we have $g(f(v_i))=g(w_i)=w'_i=f'(v'_i)=f'(h(v_i))$ for $i=1,\ldots,r$ and $g(f(v_i))=g(0)=0=f'(v'_i)=f'(h(v_i))$ for $i=r+1,\ldots,n$. Therefore, altogether $g\circ f=f'\circ h$, where the matrices $\mathfrak{C}':=\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(h)$ and $\mathfrak{B}:=\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(g)$ are invertible. It follows that $\mathfrak{B}\mathfrak{A}=\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(g)\,\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(f)=\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(g\circ f)=\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(f'\circ h)=\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(f')\,\mathfrak{M}^{\mathfrak{e}}_{\mathfrak{e}}(h)=\mathfrak{A}'\mathfrak{C}'$ and hence $\mathfrak{A}'=\mathfrak{B}\mathfrak{A}\mathfrak{C}$ mit $\mathfrak{C}:=(\mathfrak{C}')^{-1}$.
For the converse the isomorphisms g and h defined above by $\mathfrak{B}$ respectively $\mathfrak{C}^{-1}$, naturally $\operatorname{Rank}\mathfrak{A}=\operatorname{Dim}\operatorname{Im}f=\operatorname{Dim}\operatorname{Im}g\circ f=\operatorname{Dim}\operatorname{Im}f'\circ h=\operatorname{Dim}\operatorname{Im}f'=\operatorname{Rank}\mathfrak{A}'$. – **Remark:** In this case we say that $\mathfrak{A}$ and $\mathfrak{A}'$ are (rank)-equivalent. The corresponding equivalence classes are precisely the set of all matrices of same rank. Therefore the rank is the only invariant of such equivalence classes. See also Test-Exercise T8.6.)

8.7 Let $m,n\in\mathbb{N}^*$, $s:=\operatorname{Min}\{m,n\}$. For every r with $0\le r\le s$, let $\mathfrak{U}_r:=\sum_{i=1}^r\mathfrak{E}_{ii}\in\mathrm{M}_{m,n}(K)$. If $\mathfrak{A}\in\mathrm{M}_{m,n}(K)$, then $\mathfrak{A}$ (rank)-equivalent to $\mathfrak{U}_r$, where $r:=\operatorname{Rank}\mathfrak{A}$. The matrices $\mathfrak{U}_0,\ldots,\mathfrak{U}_s$ form a full representative system in $\mathrm{M}_{m,n}(K)$ with respect to the relation of equivalence of matrices given in the Exercise 8.6 above. (**Remark :** Multiplying by elementary matrices $\mathfrak{B}_{ij}(a)$, $i<j$ from right and $\mathfrak{B}_{ij}(a)$, $i>j$ from left, we can even find an invertible upper triangular matrix $\mathfrak{A}_2$ and an invertible lower triangular matrix $\mathfrak{A}_1$ such that from the matrix $\mathfrak{A}_1\mathfrak{A}\mathfrak{A}_2$ one can obtain $\mathfrak{U}_r$ by multiplying columns and rows by suitable scalars and permuting them.)

****8.8** Let K be an arbitrary field and let $\mathbf{a}:=(a_1,\ldots,a_n)\in K^n$, $n\in\mathbb{N}^+$. Let $U\subseteq K^n$ be a K-subspace of the K-vector space K^n generated by the $n!$ vectors $\mathbf{a}_\sigma:=(a_{\sigma(1)},\ldots,a_{\sigma(n)})$, $\sigma\in\mathfrak{S}_n$, obtained by permuting the coordinates of $(a_1,\ldots,a_n)$. Compute the dimension Dim_KU of U. (**Hint :** Let $\mathfrak{A}:=\big(a_{\sigma(i)}\big)_{\sigma\in\mathfrak{S}_n,\,1\le i\le n}\in\mathrm{M}_{n!\times n}(K)$ and let ${}^{\mathrm{t}}f:K^n\to K^{n!}$ be the K-linear map defined by ${}^{\mathrm{t}}f(e_i):=\mathfrak{c}_i=\sum_{\sigma\in\mathfrak{S}_n}a_{\sigma(i)}e_\sigma$, $i=1,\ldots,m$, where $e_1,\ldots,e_n\in K^n$ and $e_\sigma\in K^{n!}=K^{\mathfrak{S}_n}$, $\sigma\in\mathfrak{S}_n$ are the standard bases of K^n and $K^{n!}$, respectively and $\mathfrak{c}_i$ denote the i-th column of $\mathfrak{A}$. Then $\operatorname{Dim}_KU=\operatorname{Rank}\mathfrak{A}=\operatorname{Rank}{}^{\mathrm{t}}\mathfrak{A}=\operatorname{Rank}{}^{\mathrm{t}}f$. Now compute the kernel $\operatorname{Ker}{}^{\mathrm{t}}f$ and use the Rank-Theorem to compute $\operatorname{Rank}{}^{\mathrm{t}}f$.)

$$\textbf{Ans: }\operatorname{Dim}_KU=\begin{cases}0, & \text{if } a_1=\cdots=a_n=0\,,\\ 1, & \text{if } a_1=\cdots=a_n\neq 0\,,\\ n-1, & \text{if } a_1\neq a_2 \text{ and } \sum_{i=1}^n a_i=0\,,\\ n, & \text{if } a_1\neq a_2 \text{ and } \sum_{i=1}^n a_i\neq 0\,,\end{cases}$$

On the other side one can see auxiliary results and (simple) Test-Exercises.

Auxiliary Results/Test-Exercises

To understand and appreciate the Test-Exercises which are marked with the symbol † one may possibly require more mathematical maturity than one has! These are steps towards applications to various other branches of mathematics, especially to analysis, number theory and Affine and Projective Geometry.

T8.1 (Matrix multiplication[7]) The following (classical) example may help you to understand why multiplication of matrices is defined the way it is. One can also see the (boring) numerical example[8]. Let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \qquad & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

be a system of m linear equations in n unknowns $x_1, \ldots, x_n$ over a field K. If we make the linear (homogeneous) change of variables, (i. e. substitute the following expressions for $x_1, \ldots, x_n$)

$$\begin{aligned} x_1 &= b_{11}y_1 + b_{12}y_2 + \cdots + b_{1\ell}y_\ell \\ x_2 &= b_{21}y_1 + b_{22}y_2 + \cdots + b_{2\ell}y_\ell \\ \cdots \quad & \cdots \quad \cdots \quad \cdots \\ x_n &= b_{m1}y_1 + b_{m2}y_2 + \cdots + b_{n\ell}y_\ell \end{aligned}$$

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[7] Matrix multiplication is very different from matrix addition and subtraction. we do not multiply corresponding entries; in particular, $\begin{pmatrix} 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 & 5 \end{pmatrix} \neq \begin{pmatrix} 2\cdot 4 = 8 & 3\cdot 5 = 15 \end{pmatrix}$! Indeed, we know that these matrices are not even "compatible" for matrix multiplication. At first glance, the definition of matrix multiplication may seem strange and complicated. However, it is defined in a way that makes it perfect for working with systems of equations.

[8] The students in a large high school (grades 9 through 12) get there in a variety of ways: by bike, by bus, and by car. The percentage of students using different modes of transportation is summarized on the left below. The total number of male and female students in each grade is summarized in the table on the top right.

					Gender	Male	Female
					9th	110	105
					10th	100	95
					11th	95	90
					12th	85	80
Modes of Transportation	9th	10th	11th	12th			
Bike	25%	20%	15%	10%		$0.25\times 110 + 0.20\times 100 + 0.15\times 95 + 0.10\times 85 = 70$	$0.25\times 105 + 0.20\times 95 + 0.15\times 90 + 0.10\times 80 = 67$
Bus	55%	65%	55%	40%		$0.55\times 110 + 0.65\times 100 + 0.55\times 95 + 0.40\times 85 = 212$	$0.55\times 105 + 0.65\times 95 + 0.55\times 90 + 0.40\times 80 = 201$
Car	20%	15%	30%	50%		$0.20\times 110 + 0.15\times 100 + 0.15\times 95 + 0.30\times 85 = 108$	$0.20\times 105 + 0.15\times 95 + 0.15\times 90 + 0.30\times 80 = 102$

Now strip away the labels, record the percentages as decimals, and suppress the computations. Put the "Modes" matrix in blue and the "Gender" matrix in purple. The product of these two matrices is shown in white and is displayed in the most conventional way as:

$$\begin{pmatrix} 0.25 & 0.20 & 0.15 & 0.10 \\ 0.55 & 0.65 & 0.55 & 0.40 \\ 0.20 & 0.15 & 0.30 & 0.50 \end{pmatrix} \cdot \begin{pmatrix} 110 & 105 \\ 100 & 95 \\ 95 & 90 \\ 85 & 80 \end{pmatrix} = \begin{pmatrix} 70 & 67 \\ 212 & 201 \\ 108 & 102 \end{pmatrix}.$$

in the above system of linear equations, then we obtain the following new system of m linear equations in ℓ unknowns $y_1,\ldots,y_\ell$:

$$\begin{aligned} c_{11}y_1 + c_{12}y_2 + \cdots + c_{1\ell}y_\ell &= b_1 \\ c_{21}y_1 + c_{22}y_2 + \cdots + c_{2n}y_\ell &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad & \quad \cdots \\ c_{m1}y_1 + c_{m2}y_2 + \cdots + c_{mn}y_\ell &= b_m \end{aligned}$$

where the matrix of coefficients $\mathfrak{C} = (c_{ir})_{\substack{1\le i\le m\\ 1\le r\le \ell}} \in \mathrm{M}_{m,\ell}(K)$ is obtained by multiplying the $m\times n$-matrix of coefficients $\mathfrak{A} = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}} \in \mathrm{M}_{m,n}(K)$ with the $n\times\ell$-matrix of coefficients of the change of variables $\mathfrak{B} = (b_{jr})_{\substack{1\le j\le n\\ 1\le r\le \ell}} \in \mathrm{M}_{n,\ell}(K)$, i. e. $\mathfrak{C} = \mathfrak{A}\cdot\mathfrak{B}$, or equivalently,

$$c_{ir} = (a_{i1},\ldots,a_{in})\cdot\begin{pmatrix} b_{1r}\\ \vdots \\ b_{nr}\end{pmatrix} = \sum_{j=1}^{n} a_{ij}b_{jr} = a_{i1}b_{jr} + \cdots + a_{in}b_{nr}\,.$$

T8.2 For the following $\mathbb{K}$- linear maps find the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f) \in \mathrm{M}_{\mathbb{N},\mathbb{N}}(\mathbb{K})$ of f with respect to the basis $\mathfrak{v} := \{t^i \mid i\in\mathbb{N}\}$ of the polynomial algebra $\mathbb{K}[t]$.

1) $f:\mathbb{K}[t]\to\mathbb{K}[t]$, $x(t)\mapsto \dot{x}(t)$ (the derivative of $x(t)$ with respect to t).

2) $f:\mathbb{K}[t]\to\mathbb{K}[t]$, $x(t)\mapsto y(t)\cdot x(t)$, where $y(t) := a_0+\cdots+a_nt^n$ is a fixed polynomial in $\mathbb{K}[t]$.

3) $f:\mathbb{K}[t]\to\mathbb{K}[t]$, $x(t)\mapsto x(t+1)$.

T8.3 Let $\mathfrak{A}\in\mathrm{M}_{I,J}(K)$ and $i\in I,\ j\in J$. Compute $e_i\mathfrak{A}$ and $\mathfrak{A}e_j$, where $e_i\in K^{(I)}$ is the standard row-vector in $K^{(I)}$ and $e_j\in K^{(J)}$ is the standard column-vector in $K^{(J)}$.

T8.4 Compute the matrix product

$$\begin{pmatrix} a_1\\ \vdots\\ a_m\end{pmatrix}(b_1,\ldots,b_n)\,,$$

where $a_1,\ldots,a_m, b_1,\ldots,b_n$ are elements in a field.

T8.5 Let I,J be finite sets. For a matrix $\mathfrak{A}\in\mathrm{M}_{I,J}(K)$, compute the products $\mathfrak{E}_{ij}\mathfrak{A}$ respectively, $\mathfrak{A}\mathfrak{E}_{rs}$, where $\mathfrak{E}_{ij}\in\mathrm{M}_I(K)$ and $\mathfrak{E}_{rs}\in\mathrm{M}_J(K)$ are the elements in the standard basis of $\mathrm{M}_{I,J}(K)$.

T8.6 Let $f:V\to W$ be a K-linear map from the n-dimensional vector space into the m-dimensional vector space W. There exist bases $\mathfrak{v}=\{v_1,\ldots,v_n\}$ of V and $\mathfrak{w}=\{w_1,\ldots,w_m\}$ of W such that the matrix $\mathfrak{m}^{\mathfrak{v}}_{\mathfrak{w}}(f)$ of f with respect to $\mathfrak{v}$ and $\mathfrak{w}$ is a matrix of the form

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 1 & 0 & \cdots & 0\\ 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{M}_{m,n}(K)\,.$$

The number of 1's in this matrix is the rank of f and hence is uniquely determined. (**Hint :** As in the proof of Rank-Theorem show that there exists a basis $u_1,\ldots,u_r,v_1,\ldots,v_s$ of V such that $u_1,\ldots,u_r$ is a basis of $\operatorname{Ker} f$ and $w_1 := f(v_1),\ldots,w_s := f(v_s)$ is a basis of $\operatorname{Im} f$. Put $v_{s+j} := u_j$ for $j=1,\ldots,r$ and a basis $\mathfrak{v} := (v_1,\ldots,v_n)$, $n:=r+s$ of V. Moreover, extend $w_1,\ldots,w_s$ to a basis $\mathfrak{w} := (w_1,\ldots,w_m)$ of W. Then $f(v_j)=w_j$ for $j=1,\ldots,r$ and $f(v_j):=0$ for $j=r+1,\ldots,n$, i. e. $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{w}}(f)$ has the required form. – On the other hand if $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{w}}(f)$ has the given form with respect to some bases $\mathfrak{v}$ of V and $\mathfrak{w}$ of W, then the image $\operatorname{Im} f$ has the basis $w_1,\ldots,w_s$, where s is the number of 1's and hence $\operatorname{Rank} f = \operatorname{Dim}\operatorname{Im} f = s$.)

T8.7 Let V be a finite dimensional K-vector space and let $g\in\operatorname{End}_K(V)$ with $\operatorname{Rank}(g)=1$. Show that there exist $y\in V$ and $e\in V^*$ such that $g(x)=e(x)\cdot y$ for every $x\in V$. Further, show that

(a) The elements y, e are unique up to scalar multiples in $K^{\times}$ and the element $e(y) \in K$ is unique and we will denote it by $\lambda = \lambda(g)$. Further, show that $\lambda(g) = 0$ if and only if $g^2 = 0$.

(b) There exists a basis $\mathfrak{v} = \{v_1, \ldots, v_n\}$ of V such that the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(g)$ of g with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

according as $\lambda \neq 0$ or $\lambda = 0$.

T8.8 (Pseudo-reflections and reflections) Let V be a finite dimensional K-vector space. An automorphism $f \in \mathrm{Aut}_K(V)$ is called a pseudo-reflection of V if $\mathrm{Rank}(f - \mathrm{id}_V) = 1$. A pseudo-reflection f of V is called a dilatation (respectively transvection or shearing) if $\lambda(f - \mathrm{id}_V) \neq 0$ (respectively $\lambda(f - \mathrm{id}_V) = 0$), see Test-Exercise T8.7.

(a) For $f \in \mathrm{Aut}_K(V)$, show that the following conditions are equivalent :

(i) f is a pseudo-reflection of V. (ii) The set $\mathrm{Fix}(f) := \{x \in V \mid f(x) = x\}$ of fixed points of f is a hyperplane in V.

(iii) There exist a vector $y \in V$, $y \neq 0$ and a linear form $e \in V^*$, $e \neq 0$ on V such that $f(x) = x + e(x) \cdot y$ for every $x \in V$.

Moreover, if these equivalent conditions are satisfied then f is a dilatation (respectively transvection) according as $e(y) \neq 0$ (respectively $e(y) = 0$).

(b) Show that the inverse of a dilatation (respectively transvection) is a dilatation (respectively transvection). (**Hint :** If $f \in \mathrm{Aut}_K(V)$ is a pseudo-reflection then write f^{-1} in the form $\mathrm{id}_V + h$.)

(c) Show that every $f \in \mathrm{Aut}_K(V)$ is a product of transvections and at most one dilatation. (**Hint :** Prove by induction on $m := \mathrm{Rank}(f - \mathrm{id}_V)$. If $m \geq 2$ and $z \notin U := \mathrm{Ker}(f - \mathrm{id}_V)$, then show that there exists $f_1 \in \mathrm{Aut}_K(V)$ which is a transvection or a product of two transvections such that $f_1(z) = f(z)$ and $f_1(x) = x$ for every $x \in U$. Now consider $f_1^{-1} f$.)

(d) A pseudo-reflection $f \in \mathrm{Aut}_K(V)$ of V is called a reflection of V if $f^2 = \mathrm{id}_V$. If $\mathrm{Char}\, K = 2$, then $f \in \mathrm{Aut}_K(V)$ is a reflection of V if and only if f is a transvection of V. Suppose that $\mathrm{Char}\, K \neq 2$. For $f \in \mathrm{Aut}_K(V)$, show that the following conditions are equivalent :

(i) f is a reflection of V.

(ii) There exist a vector $y \in V, y \neq 0$ and a linear form $e \in V^*, e \neq 0$ on V such that $e(y) = -2$ and $f(x) = x + e(x) \cdot y$ for every $x \in V$.

(iii) There exists a basis $\mathfrak{v} = \{v_1, \ldots, v_n\}$ of V such that the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of f with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In particular, if f is a reflection then it is a dilatation.

T8.9 Let V be n-dimensional K-vector space and let $f \in \mathrm{End}_K(V)$. Show that (in all matrices given below, entries at the non-marked places are 0)

(a) f is a projection, i.e. $f^2 = f$ if and only if there exists a basis $\mathfrak{v} = \{v_1, \ldots, v_n\}$ of V such that the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of f with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \in \mathrm{M}_n(K).$$

(**Hint :** By Test-Exercise T7.21 f is a projection if and only if there exists a basis $\mathfrak{x} = (x_1, \ldots, x_n)$ of V such that $f(x_i) = x_i$, $i = 1, \ldots, r$, and $f(x_i) = 0$, $i = r+1, \ldots, n$, or equivalently, the matrix $\mathfrak{M}^{\mathfrak{x}}_{\mathfrak{x}}(f)$ has the required form.)

(b) Suppose that $\operatorname{Char} K \neq 2$. Then f is an involution, i.e. $f^2 = \mathrm{id}_V$ if and only if there exists a basis $\mathfrak{v} = \{v_1, \ldots, v_n\}$ of V such that the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of f with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \in \mathrm{M}_n(K).$$

(**Hint :** Note that f is an involution of V if and only if $g := \frac{1}{2}(\mathrm{id}_V - f)$ is a projection, since $g^2 = \frac{1}{4}(\mathrm{id}_V - f)^2 = \frac{1}{4}(\mathrm{id}_V - 2f + f^2) = \frac{1}{4}(2\mathrm{id}_V - 2f) + \frac{1}{4}(f^2 - \mathrm{id}_V) = g + \frac{1}{4}(f^2 - \mathrm{id}_V)$, we have $g^2 = g$ if and only if $f^2 = \mathrm{id}_V$. By the above part (a) g is a projection if and only if there exists a basis $\mathfrak{x} = (x_1, \ldots, x_n)$ of V such that $g(x_i) = x_i$, $i = 1, \ldots, r$, and $g(x_i) = 0$, $i = r+1, \ldots, n$, and hence (since $f = \mathrm{id} - 2g$) $f(x_i) = x_i - 2x_i = -x_i$, $i = 1, \ldots, r$, and $f(x_i) = x_i - 0 = x_i$, $i = r+1, \ldots, n$, or equivalently, the matrix of f with respect to the basis $x_n, \ldots, x_1$ has the required form.)

(c) Show that f is a transvection (see Test-Exercise T8.8) if and only if there exists a basis $\mathfrak{v} = \{v_1, \ldots, v_n\}$ of V such that the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of f with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} 1 & 0 & & & \\ 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in \mathrm{M}_n(K).$$

(**Hint :** By definition f is a transvection if the fixed-point space of f is the hyperplane $\mathrm{Fix}(f) := H := \{x \in V \mid f(x) = x\}$ and if for one (and hence for every) $x \in V \setminus H$ the direction of the reflection $f(x) - x$ belongs to H. In this case choose a $v_1 \in V - H$ and extend $v_2 := f(x) - x \neq 0$ by adding $v_3 \ldots, v_n$ to a basis $v_2, \ldots, v_n$ of H. Then $\mathfrak{v} = (v_1, \ldots, v_n)$ is a basis of V, and we have $f(v_1) = v_1 + (f(v_1) - v_1) = v_1 + v_2$ and $f(v_j) = v_j$ for $j = 2, \ldots, n$. $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ has the required form. Conversely, if f has such a matrix representation with respect to a basis $v_1, \ldots, v_n$, then $(f(v_j) = v_j$ for all $j = 2, \ldots, n$, and $f(v_1) = v_1 + v_2$, and hence $H := \{x \in V \mid f(x) = x\}$ $(n-1)$-dimensional and for $x := v_1 \notin H$ we have $f(x) - x = v_2 \in H$, i. e. f is a transvection.)

(d) Show that f is a dilatation (see Test-Exercise T8.8) if and only if there exists a basis $\mathfrak{v} = \{v_1, \ldots, v_n\}$ of V such that the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of f with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathrm{M}_n(K)$$

where $\lambda \in K, \lambda \neq 0, 1$. (**Hint :** By definition an automorphism f is a dilatation, if the fixed-point space of f is the hyperplane $\mathrm{Fix}(f) := H := \{x \in V \mid f(x) = x\}$ is a hyperplane in V and if for one (and hence for every) $x \in V \setminus H$ the direction of the reflection $f(x) - x$ does not lie in H. In this case we put $v_1 := f(x) - x$ and extend v_1 to a basis $v_1, v_2, \ldots, v_n$ of H. Then $f \upharpoonright H = \mathrm{id}_H$, and $f(x) = \lambda x + h$ with $h \in H$, and so $\lambda \neq 0$, since otherwise $\mathrm{Im} f \subseteq H$ a contradiction to the bijectivity of f, and further $\lambda \neq 1$, since otherwise $f(x) - x \in H$. Moreover, $f(v_1) = f(f(x) - x) = f(\lambda x + h) - f(x) = \lambda f(x) + h - (\lambda x + h) = \lambda(f(x) - x) = \lambda v_1$ and $f(v_i) = v_i$ for $i = 2, \ldots, n$. Therefore $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ has a required form. The converse is trivial.)

T8.10 Let V be an n-dimensional K-vector space and let $f : V \to V$ be a linear operator. Then the matrices $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ and $\mathfrak{M}^{\mathfrak{v}'}_{\mathfrak{v}'}(f)$ of f with respect to bases $\mathfrak{v}$ and $\mathfrak{v}'$ of V, respectively, are equal if and only if f is a homothecy $a\,\mathrm{id}_V$, $a \in K$.

T8.11 From the above Test-Exercise T8.8 deduce that : for a finite dimensional K-vector space V:

(a) The center $\mathrm{Z}(\mathrm{End}_K V)$ of the K-algebra $\mathrm{End}_K V$, i.e. the subalgebra of $f \in \mathrm{End}_K V$ which commute with all elements of $\mathrm{End}_K V$ is equal to the subalgebra $\{a\,\mathrm{id}_V \mid a \in K\}$ of homothecies of V.

(b) The center $\mathrm{Z}(\mathrm{Aut}_K V)$ of the automorphism group $\mathrm{Aut}_K V$ of V is the subgroup $\{a\,\mathrm{id}_V \mid a \in K^\times\}$ of homothecies of V.

(c) What is the center of the matrix algebra $\mathrm{M}_I(K)$ resp. the group $\mathrm{GL}_I(K)$? where I is a finite set.

T8.12 Let V be K-vector space of dimensions n, $\mathfrak{v} = \{u_1, \ldots, u_r, w_1, \ldots, w_s\}$ be a K-basis of V, $U := Ku_1 + \cdots + Ku_r$, $W := Kw_1 + \cdots Kw_s$ and let $f \in \mathrm{End}_K(V)$. Then

(a) The subspace U of V is *invariant under* f, i.e. $f(U) \subseteq U$ if and only if the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of f with respect to $\mathfrak{v}$ is of the form

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} & c_{11} & \cdots & c_{1s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} & c_{r1} & \cdots & c_{rs} \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{s1} & \cdots & b_{ss} \end{pmatrix} \in \mathrm{M}_{r+s}(K).$$

In this case

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix} \in \mathrm{M}_r(K) \qquad \text{and} \qquad \begin{pmatrix} b_{11} & \cdots & b_{1s} \\ \vdots & \ddots & \vdots \\ b_{s1} & \cdots & b_{ss} \end{pmatrix} \in \mathrm{M}_s(K)$$

is the matrix of $f|U$ with respect to the basis $\mathfrak{u} = \{u_1 \ldots, u_r\}$ of U resp. the matrix of the K-linear map $\overline{f} : V/U \to V/U$ induced by $\bar{f}$ with respect to the (residue class-)basis $\overline{\mathfrak{w}} =$ $overline w_1, \ldots, \overline{w_s}\}$ of $\overline{V} := V/U$.

(b) Both the subspaces U and W of V are invariant under f, i.e. $f(U) \subseteq U$ and $f(W) \subseteq W$ if and only if $c_{ij} = 0$ for all $1 \le i \le r, 1 \le j \le s$ in the matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of the part a).

T8.13 The matrix $\mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}}(f)$ of the part a) is usually written as the block matrix $\begin{pmatrix} \mathfrak{A} & \mathfrak{C} \\ 0 & \mathfrak{B} \end{pmatrix}$, where $\mathfrak{A} \in \mathrm{M}_r(K), \mathfrak{B} \in \mathrm{M}_s(K), \mathfrak{C} \in \mathrm{M}_r(K)$. Show that such a block matrix is invertible if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are invertible. Further, show that

$$\begin{pmatrix} \mathfrak{A} & \mathfrak{C} \\ 0 & \mathfrak{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathfrak{A}^{-1} & -\mathfrak{A}^{-1}\mathfrak{C}\mathfrak{B}^{-1} \\ 0 & \mathfrak{B}^{-1} \end{pmatrix}.$$

T8.14 The matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{M}_2(K)$ is invertible if and only if $ad - bc \neq 0$. Its inverse is then

$$\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

T8.15 Find the matrix of the linear map $f : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$f(a_1, a_2) := (3a_1 + 3a_2\,, 2a_1 - a_2\,, -5a_1 + 3a_2\,, 4a_1 - 3a_2)$$

with respect to the standard bases of $\mathbb{R}^2$ respectively $\mathbb{R}^4$; also find it with respect to the bases $(1,1)$, $(1,2)$ of $\mathbb{R}^2$ respectively $(1,0,0,1)$, $(0,1,1,0)$, $(0,0,1,1)$, $(0,0,1,0)$ of $\mathbb{R}^4$.

T8.16 Suppose that the endomorphism f of $\mathbb{Q}^3$ have the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

with respect to the standard basis e_1, e_2, e_3 of $\mathbb{Q}^3$. Show that the vectors $e_1 - e_2 - e_3$, $2e_2 - e_3$, $e_1 + e_2$ form a basis of $\mathbb{Q}^3$. Moreover, find the matrix of f with respect to this basis of $\mathbb{Q}^3$.

T8.17 Let I be a finite set. The map $f : \mathrm{GL}_I(K) \to \mathrm{GL}_I(K)$ defined by $\mathfrak{A} \mapsto {}^t\mathfrak{A}^{-1}$, (which maps every matrix to its contra-gredient matrix) is an automorphism of the group $\mathrm{GL}_I(K)$. Moreover, its inverse is itself. (**Hint :** $f(\mathfrak{A}\mathfrak{B}) = {}^t(\mathfrak{A}\mathfrak{B})^{-1} = {}^t(\mathfrak{B}^{-1}\mathfrak{A}^{-1}) = {}^t\mathfrak{A}^{-1}\,{}^t\mathfrak{B}^{-1} = f(\mathfrak{A})\,f(\mathfrak{B})$. Further, $f\big(f(\mathfrak{A})\big) = {}^t\big({}^t\mathfrak{A}^{-1}\big)^{-1} = \big(\mathfrak{A}^{-1}\big)^{-1} = \mathfrak{A}$ and hence $f^2 = \mathrm{id}$.)

T8.18 In $\mathrm{M}_n(K)$, for all $a \in K^\times$ and all $m \in \mathbb{Z}$, prove that

$$\begin{pmatrix} a & 1 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 1 \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}^m = \begin{pmatrix} a^m & \binom{m}{1}a^{m-1} & \cdots & \binom{m}{n-2}a^{m-n+2} & \binom{m}{n-1}a^{m-n+1} \\ 0 & a^m & \cdots & \binom{m}{n-3}a^{m-n+3} & \binom{m}{n-2}a^{m-n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^m & \binom{m}{1}a^{m-1} \\ 0 & 0 & \cdots & 0 & a^m \end{pmatrix}.$$

(**Hint :** We denote by $\mathfrak{D}_{n,1} := (\delta_{i+1,j})_{1\le i,j\le n} = (\delta_{i,j-1})_{1\le i,j\le n} \in \mathrm{M}_n(K)$ the $(n\times n)$-matrix, in which the first next-diagonal above the main-diagonal has 1 everywhere and all other coefficients are 0. More generally, we put $\mathfrak{D}_{n,k} := (\delta_{i+k,j})_{1\le i,j\le n} \in \mathrm{M}_n(K)$ the $(n\times n)$-matrix, in which k-th next-diagonal above the main-diagonal has 1 everywhere and all other coefficients are 0 everywhere. Then $\mathfrak{D}_{n,0} = \mathfrak{E}_n$ the identity matrix, and for $k \in \mathbb{N}$, we have $(\mathfrak{D}_{n,1})^k = \mathfrak{D}_{n,k}$. From this the inductive -step from k to $k+1$ follows, since the element in the i-th row and the ℓ-th column of $\mathfrak{D}_{n,1})^{k+1} = (\mathfrak{D}_{n,1})^k\,\mathfrak{D}_{n,1} = \mathfrak{D}_{n,k}\mathfrak{D}_{n,1}$ is equal to $\sum_{j=1}^{n} \delta_{i+k,j}\,\delta_{j,\ell-1} = \delta_{i+k,\ell-1} = \delta_{i+k+1,\ell}$, which is also the corresponding element of $\mathfrak{D}_{n,k+1}$. In particular, it follows that $\mathfrak{D}_{n,1}^n = \mathfrak{D}_{n,n} = 0$. Now, the m-th power of the matrix $a\,\mathfrak{E}_n + \mathfrak{D}_{n,1}$ is: $(a\,\mathfrak{E}_n + \mathfrak{D}_{n,1})^m = \sum_{k=1}^{n} \binom{m}{k} a^{m-k}(\mathfrak{E}_n)^{m-k}(\mathfrak{D}_{n,1})^k = \sum_{k=1}^{n} \binom{m}{k} a^{m-k}\mathfrak{D}_{n,k}$. This is precisely the given matrix on the right-hand side.)

T8.19 In $\mathrm{M}_n(K)$ with $n-1 \in K^\times$, prove that

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{n-1}\begin{pmatrix} 1 & 1 & \cdots & 1 & 2-n \\ 1 & 1 & \cdots & 2-n & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2-n & \cdots & 1 & 1 \\ 2-n & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

(**Hint :** It is enough to show that the product $\big(1-\delta_{n-i+1,j}\big)_{1\le i,j\le n} \cdot \big(\tfrac{1}{n-1} - \delta_{j,n-\ell+1}\big)_{1\le j,\ell\le n}$ is the identity matrix. This follows from the fact that in the i-th row and ℓ-th column of the product of these matrices is the following element:

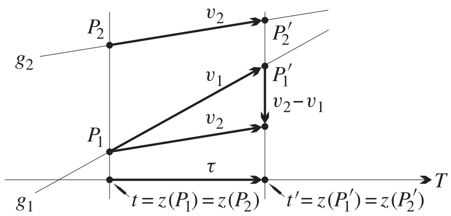
$$\begin{aligned}\sum_{j=1}^{n}(1-\delta_{n-i+1,j})\left(\frac{1}{n-1} - \delta_{j,n-\ell+1}\right) &= \sum_{j=1}^{n}\frac{1}{n-1} - \frac{1}{n-1}\sum_{j=1}^{n}\delta_{n-i+1,j} - \sum_{j=1}^{n}\delta_{j,n-\ell+1} + \sum_{j=1}^{n}\delta_{n-i+1,j}\,\delta_{j,n-\ell+1} \\ &= \frac{n}{n-1} - \frac{1}{n-1} - 1 + \delta_{i\ell} = \delta_{i\ell}\,.)\end{aligned}$$

T8.20 Let $\mathfrak{v} = (v_i)_{i\in I}$ and $\mathfrak{v}' = (v_i)_{i\in I}$ be bases of the finite dimensional K-vector space V and let $\mathfrak{v}^*$ resp. $\mathfrak{v}'^*$ be the corresponding dual bases of V^*. If $\mathfrak{A} = \mathfrak{M}^{\mathfrak{v}}_{\mathfrak{v}'}(\mathrm{id}_V)$ is the transition matrix from the basis $\mathfrak{v}$ to the basis $\mathfrak{v}'$, then show that the contra-gradient matrix ${}^t\mathfrak{A}^{-1}$ is the transition matrix $\mathfrak{M}^{\mathfrak{v}'^*}_{\mathfrak{v}^*}(\mathrm{id}_{V^*})$ from the basis $\mathfrak{v}^*$ to the basis $\mathfrak{v}'^*$.

†**T8.21** (Classical space-time-world) Perhaps the greatest obstacle to understand the theories of *special* and *general relativity* [9] arises from the difficulty in realising that a number of previously held basic assumptions about the nature of space and time are wrong. We therefore spell-out some key assumptions

[9] The general theory of relativity is one of the greatest intellectual achievements of all time. Its originality and unorthodox approach exceed that of special relativity. And for so more than special relativity, it was almost completely the work of a single man, Albert Einstein (1879-1955). The philosophic impact of relativity theory on the thinking of man has been profound and the vistas of science opened by it are literally endless.

about space and time. We can consider space and time ($\equiv$ space-time [10]) to be a continuum composed of events, where each event can be thought as a point of space at an instant of time.
Up to now we have only considered the *universe* S over the vector space V_S of translations, and time was ignored. Classically, time is a real affine line T. The corresponding vector space is denoted by V_T; for the measurement of time, we choose a basis τ of V_T, pointing into the "future", i.e. for given moments t_1 and t_2 in T, we say that "t_1 comes before t_2" if the vector $\overrightarrow{t_1t_2}$ has a representation $a\tau$ with a positive real number a (arrow in the direction of time). The motion of a free particle on a line in the universe gives an isomorphism of this line onto T. The most naive description of the space-time-world as a whole is done through the four-dimensional product space $S\times T$ which is, in a natural way, an affine space over the $\mathbb{R}$-vector space $V_S\times V_T$. Both the projections of $S\times T$ onto S and T are affine maps. They associate to every world-point in $S\times T$ its position resp. its time. The fibres of these projections are the points with the same position resp. time.
It has been known from early times – at least from the time of Aristotle – that it does not make sense to talk about two events taking place at different times at the same place. Description of position is only possible relative to a frame of reference; one cannot distinguish any one of these frames of reference as a fixed frame of reference. On the other hand, in the area of classical physics one has the concept of simultaneousness: Two distinct world-points are not simultaneous if and only if (at least in the mental experiment) the same mass-point can occupy both these world-points.
Therefore one describes the classical space-time-world as a four dimensional real affine space E with an affine (non-constant) map $z:E\to T$ from E onto the time T. For an event $P\in E$, we call $z(P)$ the time at which the event P takes place. The fibres of the affine map z define the space-directions. Our universe, which we have handled so far, was always such a fibre. All these fibres are parallel to the three-dimensional subspace V_S of the vector space V_E corresponding to E.
Two world-points P and Q in E differ from each other by the vector $\overrightarrow{PQ}$. P and Q are simultaneous if and only if $\overrightarrow{PQ}\in V_S$. Therefore the vectors in V_S are called space-like vectors. Every vector in V_E, which is not a space-like vector, is called time-like. The world-points representing the motion of a free particle m_1 (which is not subject to any outer forces), form an affine line $g_1=\mathbb{R}v_1+P_1$ in E, the so called world-line of these mass-points. It is parallel to the line $\mathbb{R}v_1$ in V_E generated by some time-like vector v_1 (Galilean law of inertia). Then the line g_1 representing the time and the affine subspace V_S+P_1 give a decomposition of E into space and time (as above) . After normalising the vector v_1 by the condition $z_0(v_1)=\tau$, where z_0 is the linear part of z, this vector v_1 is called the absolute or four-velocity of the mass-point under consideration.
If m_2 is another mass-point with the absolute velocity v_2 (moving freely without being subject to outer forces) , then $v_2-v_1\in V_S$ is a space-like vector. It is called the relative velocity of m_2 with respect to m_1.



The simultanousness
as defined above requires arbitrary large relative velocities. Since observations suggest that arbitrary large velocities cannot occur, one tries to abandon the notion of simultanousness. A first step in this direction is the special theory of relativity.
As automorphisms of the classical space-time-world E described above we shall consider the affinities f of E,

[10] Hermann Minkowski (1864-1909) referred to space-time as the world, hence events are world-points and a collection of events giving history of a particle is a world-line. Physical laws on the interaction of particles can be thought of as the geometric relation between the world-lines. In this sense Minkowski maty be said to have *geometrized* physics.

which are compatible with the time map $z:E\to T$. By this we mean that there exists an affinity $f_T:T\to T$ (which is necessarily uniquely determined) such that $z\circ f=f_T\circ z$:

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle z} \\ T & \xrightarrow{\ f_T\ } & T \end{array}$$

These automorphisms f of E form a subgroup G of the affine group $\mathrm{A}(E)$ of E. This subgroup G is called the *affine Galilean group*. An affinity f in $\mathrm{A}(E)$ belongs to G if and only if its linear part maps the vector space V_S of the space-like vectors into itself. By G_0 we denote the subgroup of automorphisms h of V_E with $h(V_S)\subseteq V_S$. Then the map $G\to G_0$ defined by $f\mapsto f_0$ is a surjective group homomorphism, and its kernel is the group $\mathrm{T}(E)$ of all translations of E. In particular, $G/\mathrm{T}(E)\cong G_0$.
Sometimes the subgroup of all $f\in G$ such that the time-part f_T is the identity, is also called the affine Galilean group.

T8.22 With the notations and concepts as in the above Test-Exercise T8.21, let v_1 a time-like vector and let v_2,v_3,v_4 be a basis of the space V_S the space-like vectors. Then show that the affinity f of the space-time-world E belongs to the affine Galilei-group G if and only if its linear part f_0 with respect to the basis $v_1,\ldots,v_4$ of V_E is a block-matrix of the form

$$\begin{pmatrix} a & 0 \\ \mathfrak{c} & \mathfrak{B}\end{pmatrix},\quad a\in\mathbb{R}^\times,\ \mathfrak{B}\in\mathrm{GL}_3(\mathbb{R}),\ \mathfrak{c}\in\mathbb{R}^3=\mathrm{M}_{3,1}(\mathbb{R}),$$

Further, it preserves the time-orientation if and only if $a>0$.

T8.23 Let $\mathfrak{A},\mathfrak{B}\in\mathrm{GL}_n(\mathbb{R})$ be inverses of each other with all coefficients are ≥ 0. Then show that every row and every column of $\mathfrak{A}$ and $\mathfrak{B}$ has only one non-zero coefficient. (**Remark :** Geometrically the hypothesis mean: $\mathfrak{A}$ and $\mathfrak{A}^{-1}$ maps the cone $\mathbb{R}^n_+\subseteq\mathbb{R}^n$ into itself.)

T8.24 **(a)** Compute the rank of the following matrices over $\mathbb{Q}$:

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \\ 3 & 4 & 5 \end{pmatrix},\quad \begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 6 & -4 \\ 5 & 10 & 10 & -5 \\ 3 & 6 & 6 & 3 \end{pmatrix},\quad \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}.$$

(b) Let K be an arbitrary field. Compute the rank of the 4×4 matrix (magic-square) given in the Exercise 6.6 depending on the characteristic $\mathrm{Char}\,K$ of K. Further, compute the rank of the following $n\times n$-matrix:

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ n+1 & n+2 & \ldots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \ldots & n^2 \end{pmatrix}.$$

T8.25 Compute the rank of the matrices $\mathfrak{A},\mathfrak{B},\mathfrak{A}\mathfrak{B},\mathfrak{B}\mathfrak{A}$ over $\mathbb{Q}$ for

$$\mathfrak{A}:=\begin{pmatrix} -2 & 0 & -5 & 0 & 1 & 6 \\ -1 & 2 & 2 & 2 & 2 & 0 \\ 4 & -2 & -2 & 1 & 1 & 0 \\ 2 & 0 & 4 & -2 & 5 & 3 \\ 0 & 3 & 6 & -2 & 5 & 4 \end{pmatrix},\quad \mathfrak{B}:=\begin{pmatrix} 2 & 2 & 1 & 1 & 1 \\ 0 & 3 & -2 & 1 & 2 \\ 4 & 1 & 4 & 1 & 0 \\ 3 & -1 & 0 & -4 & 3 \\ 5 & 1 & 1 & -3 & 4 \\ 0 & -1 & -1 & -2 & 1 \end{pmatrix}.$$

T8.26 Prove the assertion on the ranks of matrices corresponding to the assertions on the ranks of linear maps given in Test-Exercises T6.16 and T6.17: For matrices $\mathfrak{A}\in\mathrm{M}_{m\times n}(K)$, $\mathfrak{B}\in\mathrm{M}_{n\times\ell}(K)$ and $\mathfrak{C}\in\mathrm{M}_{\ell\times p}(K)$, show that

(a) (S y l v e s t e r ' s i n e q u a l i t y) $\operatorname{Rank}\mathfrak{A} + \operatorname{Rank}\mathfrak{B} - n \le \operatorname{Rank}\mathfrak{A}\mathfrak{B} \le \min\{\operatorname{Rank}\mathfrak{A}, \operatorname{Rank}\mathfrak{B}\}$.

(b) (F r o b e n i u s i n e q u a l i t y) $\operatorname{Rank}\mathfrak{A}\mathfrak{B} + \operatorname{Rank}\mathfrak{B}\mathfrak{C} \le \operatorname{Rank}\mathfrak{B} + \operatorname{Rank}\mathfrak{A}\mathfrak{B}\mathfrak{C}$.

T8.27 Determine which of the following matrices are invertible over $\mathbb{Q}$ and in the appropriate cases compute the inverse matrix:

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & 2 & 3 & -4 \\ 8 & 3 & 5 & 7 \\ 7 & 2 & 4 & 6 \\ 6 & 2 & 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ 1 & 6 & -4 & -3 \end{pmatrix}.$$

T8.28 Determine which of the following matrices are invertible over $\mathbb{C}$ and in the appropriate cases compute the inverse matrix:

$$\begin{pmatrix} 1 & 0 & 1+\mathrm{i} \\ 0 & 1 & \mathrm{i} \\ 1-\mathrm{i} & -\mathrm{i} & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & \mathrm{i} \\ 1 & -3 & -\mathrm{i} \\ \mathrm{i} & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \mathrm{i} & -\mathrm{i} \\ 2\mathrm{i}-1 & 2+\mathrm{i} & \mathrm{i} \\ \mathrm{i} & 1+\mathrm{i} & 0 \end{pmatrix}.$$

T8.29 Let I, J be finite sets and let $\mathfrak{A} \in \mathrm{M}_{I,J}(K)$.

(a) For every sub-matrix $\mathfrak{U}$ of $\mathfrak{A}$, $\operatorname{Rank}\mathfrak{U} \le \operatorname{Rank}\mathfrak{A}$.

(b) The rank of $\mathfrak{A}$ is the maximum of the ranks of the invertible square sub-matrices of $\mathfrak{A}$. In particular, if $\mathfrak{A} = (a_{ij})$, $r := \operatorname{Rank}\mathfrak{A}$, then there is an injective maps σ, τ from $[1, r]$ into I resp. J such that $(a_{\sigma(i)\tau(j)})_{1\le i\le r, 1\le j\le r}$ is invertible.

(c) Let K be a subfield of the field L. Then show that $\operatorname{Rank}_K\mathfrak{A} = \operatorname{Rank}_L\mathfrak{A}$. (**Hint :** see Test-Exercise T7.36 (a)) Further, show that $\mathfrak{A}$ is invertible over K if and only if $\mathfrak{A}$ is invertible over L. (**Remark :** Naturally, then the inverses over K and over L are same.)

T8.30 Prove the Theorem 8.B.3 by using the Test-Exercise T8.6 : *Let I and J be finite sets and let* $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_{I,J}(K)$ *be an* $I \times J$*-matrix. Then* $\operatorname{Rank}\mathfrak{A} = \operatorname{Rank}{}^t\mathfrak{A}$, i. e. *the column-rank of* $\mathfrak{A}$ *is equal to the row-rank of* $\mathfrak{A}$. **Proof:** Let $f\colon K^J \to K^I$ be the linear map defined by $f(\mathfrak{x}) = \mathfrak{A}\mathfrak{x}$. Then $\mathfrak{A}$ is the matrix of f with respect to the standard bases of K^J respectively K^I. By Test-Exercise T8.6 there exist bases $\mathfrak{v}$ of K^J and $\mathfrak{w}$ of K^I such that the matrix $\mathfrak{D}$ with respect to these bases have all zero coefficients except 1's on the first r places on the main-diagonal, where $r = \operatorname{Rank} f = \operatorname{Rank}\mathfrak{A}$. If $\mathfrak{B}$ respectively $\mathfrak{C}$ are the corresponding transition matrices (with $\mathfrak{v}$ as columns of $\mathfrak{B}$ and $\mathfrak{w}$ as columns of $\mathfrak{C}$), then $\mathfrak{D} = \mathfrak{C}^{-1}\mathfrak{A}\mathfrak{B}$ by 8.A.14 and it follows from 8.A.18 and 8.A.19 that $\mathfrak{D} = {}^t\mathfrak{D} = {}^t\mathfrak{B}\,{}^t\mathfrak{A}\,{}^t\mathfrak{C}^{-1}$, i. e. $\mathfrak{D}$ and ${}^t\mathfrak{A}$ describes the same linear map $K^I \to K^J$, (only with respect to different bases). Once again it follows from the Test-Exercise T8.6 that $r = \operatorname{Rank}{}^t\mathfrak{A}$.

T8.31 Let $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$. Show that $\operatorname{Rank}\mathfrak{A} \le r$ if and only if there exist an $m \times r$-matrix $\mathfrak{B}$ and an $r \times n$-matrix $\mathfrak{C}$ over K such that $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$. Further, show that the following statements are equivalent:
(i) $\operatorname{Rank}\mathfrak{A} = r$, i. e. $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$ is a rank-factorisation of $\mathfrak{A}$. (ii) $\operatorname{Rank}\mathfrak{B} = \operatorname{Rank}\mathfrak{C} = r$. (iii) Columns of $\mathfrak{B}$ form a basis of the column-space of $\mathfrak{A}$. (iv) Rows of $\mathfrak{C}$ form a basis of the row-space of $\mathfrak{A}$.
Formulate the case $r = 1$ explicitly.

(**Remark:** For a matrix $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$ of rank $r \ge 1$, $(\mathfrak{B}, \mathfrak{C})$ is said to be a r a n k - f a c t o r i s a t i o n of $\mathfrak{A}$ if $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$ and $\mathfrak{B} \in \mathrm{M}_{m,r}(K)$ and $\mathfrak{C} \in \mathrm{M}_{r,n}(K)$. This exercise show that every non-zero matrix has a rank-factorisation. But it is not unique in general, for instance if $(\mathfrak{B}, \mathfrak{C})$ is a rank-factorisation of $\mathfrak{A}$, then for every $\mathfrak{G} \in \mathrm{GL}_r(K)$, $(\mathfrak{B}\mathfrak{G}, \mathfrak{G}^{-1}\mathfrak{C})$ is also a rank-factorisation of $\mathfrak{A}$. However, if $(\mathfrak{B}, \mathfrak{C})$ and $(\mathfrak{B}, \mathfrak{C}')$ are rank-factorisations of $\mathfrak{A}$, then $\mathfrak{C} = \mathfrak{C}'$ and similarly, if $(\mathfrak{B}, \mathfrak{C})$ and $(\mathfrak{B}', \mathfrak{C})$ are rank-factorisations of $\mathfrak{A}$, then $\mathfrak{B} = \mathfrak{B}'$.)

T8.32 Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \in K^n$ be column-vectors. Show that the $n \times n$-matrices $\mathfrak{a}_i\,{}^t\mathfrak{a}_j \in \mathrm{M}_n(K)$, $1 \le i, j \le n$, form a K-basis of $\mathrm{M}_n(K)$, if and only if $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, is a K-basis of K^n.

†**T8.33** Let $P_j = (a_{1j}, \ldots, a_{mj})$, $j = 1, \ldots, n$, be points in the affine space $\mathbb{A}^m(K) = K^m$. The dimension of the affine subspace of $\mathbb{A}^m(K)$ generated by the points $P_1, \ldots, P_n$ is 1 less than the rank of the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathrm{M}_{m+1,n}(K).$$

T8.34 Suppose that the solution spaces of the system of linear equations $\mathfrak{A}\mathfrak{x} = \mathfrak{b}$ and $\mathfrak{A}'\mathfrak{x} = \mathfrak{b}'$ with $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$, $\mathfrak{A}' \in \mathrm{M}_{m',n}(K)$ and column-vectors $\mathfrak{b} \in K^m$, $\mathfrak{b}' \in K^{m'}$, $m, m', n \in \mathbb{N}$, are non-empty affine spaces of K^n. Show that these subspaces are parallel if and only if the block-matrix $\begin{pmatrix} \mathfrak{A} \\ \mathfrak{A}' \end{pmatrix} \in \mathrm{M}_{m+m',n}(K)$ is of rank Max $(\mathrm{Rank}\,\mathfrak{A}, \mathrm{Rank}\,\mathfrak{A}')$.

T8.35 Let $r \in \mathbb{N}^*$, $s \in \mathbb{N}$ and let $\mathfrak{B} \in \mathrm{M}_s(K)$. Show that for every matrix $\mathfrak{A} \in \mathrm{M}_{s,r}(K)$ and every column-vector $\mathfrak{x} \in K^r$ there exists a column-vector $\mathfrak{y} \in K^s$ with $\mathfrak{A}\mathfrak{x} + \mathfrak{B}\mathfrak{y} = 0$, if and only if $\mathfrak{B}$ is invertible. Moreover, in this case, one can choose $\mathfrak{y} = -\mathfrak{B}^{-1}\mathfrak{A}\mathfrak{x}$.

T8.36 Let $r, s \in \mathbb{N}$ with $1 \leq r, s \leq m$ and $r \neq s$. For all a, b in the field K, show that $\mathfrak{B}_{rs}(a+b) = \mathfrak{B}_{rs}(a)\,\mathfrak{B}_{rs}(b)$ in $\mathrm{M}_m(K)$, i.e. the map $(K, +) \to \mathrm{GL}_m(K)$, $a \mapsto \mathfrak{B}_{rs}(a)$ is an (injective) homomorphism of the additive group of K in the group $\mathrm{GL}_m(K)$ of the invertible matrices. (**Hint :** $(\mathfrak{E}_n + a\mathfrak{E}_{rs})(\mathfrak{E}_n + b\mathfrak{E}_{rs}) = \mathfrak{E}_n + b\mathfrak{E}_{rs} + a\mathfrak{E}_{rs} + ab\mathfrak{E}_{rs}\mathfrak{E}_{rs} = \mathfrak{E}_n + (a+b)\mathfrak{E}_{rs}$, since $\mathfrak{E}_{rs}\mathfrak{E}_{rs} = \delta_{rs}\mathfrak{E}_{rs} = 0$.)

T8.37 Let $1 \leq j \leq m$ and let $a_{j+1}, \ldots, a_m \in K$. Show that the elementary matrices $\mathfrak{B}_{j+1,j}(a_{j+1}), \ldots, \mathfrak{B}_{m,j}(a_m) \in \mathrm{M}_m(K)$ are pairwise commutative and their product $\mathfrak{B}_{j+1,j}(a_{j+1}) \cdots \mathfrak{B}_{m,j}(a_m)$ is the normalized upper triangular matrix $\mathfrak{B}_j(a_{j+1}, \ldots, a_m)$ which is obtained from the identity matrix by replacing j-th column by adding the elements $a_{j+1}, \ldots, a_m$ under the main-diagonal, i.e. $\mathfrak{B}_j(a_{j+1}, \ldots, a_m) = \mathfrak{E}_n + \sum_{k=1}^{m-j} a_{j+k}\mathfrak{E}_{j+k,j}$. The map $(a_{j+1}, \ldots, a_m) \longmapsto \mathfrak{B}_j(a_{j+1}, \ldots, a_m)$ is a homomorphism of from the additive group K^{m-j} into the group $\mathrm{GL}_m(K)$. In particular, $\mathfrak{B}_j(a_{j+1}, \ldots, a_m)^{-1} = \mathfrak{B}_j(-a_{j+1}, \ldots, -a_m)$. (**Remark :** In the concrete situation it is practical for the row-operations to pre-multiply by the matrices of the type $\mathfrak{B}_j(a_{j+1}, \ldots, a_m)$. Similarly for column-operations.)

T8.38 The normalised lower (respectively, upper) triangular matrices
$\mathrm{LT}_n(K) := \{(a_{ij}) \in \mathrm{M}_n(K) \mid a_{ij} = 0$ for all $i < j$ and $a_{ii} = 1$ for all $i = 1, \ldots, n\}$ (respectively, $\mathrm{UT}_n(K) := \{(a_{ij}) \in \mathrm{M}_n(K) \mid a_{ij} = 0$ for all $i > j$ and $a_{ii} = 1$ for all $i = 1, \ldots, n\}$ in $\mathrm{M}_n(K)$ form a subgroup of $\mathrm{GL}_n(K)$.

T8.39 The center of the group $\mathrm{GL}_n(K)$ is the subgroup $K^\times\mathfrak{E}_n = \{a\mathfrak{E}_n \mid a \in K^\times\}$, where $\mathfrak{E}_n$ is the unit matrix. (**Hint :** Use $\mathfrak{A}\mathfrak{B}_{rs}(1) - \mathfrak{B}_{rs}(1)\mathfrak{A} = \mathfrak{A}\mathfrak{E}_{rs} - \mathfrak{E}_{rs}\mathfrak{A}$ for $1 \leq r, s \leq n$ with $r \neq s$. See also Test-Exercise T8.11-(c).)

T8.40 Let r, s, i be pairwise distinct indices in $\{1, \ldots, n\}$ and let $a \in K$. Then in $\mathrm{GL}_n(K)$ show that $\mathfrak{P}_{rs}\mathfrak{B}_{is}(a) = \mathfrak{B}_{ir}(a)\mathfrak{P}_{rs}$, $\mathfrak{P}_{rs}\mathfrak{B}_{rs}(a) = \mathfrak{B}_{sr}(a)\mathfrak{P}_{rs}$.

T8.41 Let $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$ be a $m \times n$-matrix of rank m.

(a) Show that there exists elementary matrices $\mathfrak{C}_1, \ldots, \mathfrak{C}_q \in \mathrm{M}_n(K)$ and a diagonal matrix $\mathfrak{D} = \mathrm{Diag}\,(d, 1, \ldots, 1) \in \mathrm{M}_{m,n}(K)$ such that $\mathfrak{A}\mathfrak{C}_1 \cdots \mathfrak{C}_q = \mathfrak{D}$.

(b) Show that there exists a normalised lower triangular matrix $\mathfrak{L} \in \mathrm{M}_m(K)$, a normalised upper triangular matrix $\mathfrak{R}' \in \mathrm{M}_{m,n}(K)$, a diagonal matrix $\mathfrak{D} = \mathrm{Diag}\,(d_1, \ldots, d_m) \in \mathrm{GL}_m(K)$ and a permutation matrix $\mathfrak{P}_\varphi \in \mathrm{M}_n(K)$ such that $\mathfrak{A}\mathfrak{P}_\varphi = \mathfrak{L}\mathfrak{D}\mathfrak{R}'$. (**Hint :** Analogous to 8.C.8 respectively, 8.C.9.)

T8.42 Let $\mathfrak{A} = (a_{ij}) \in \mathrm{GL}_n(K)$. For $k = 1, \ldots, n$, let

$$\mathfrak{A}_k := \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}.$$

Show that there exist a lower triangular matrix $\mathfrak{L}$ and an upper triangular matrix $\mathfrak{R}$ in $\mathrm{GL}_n(K)$ such that $\mathfrak{A} = \mathfrak{L}\mathfrak{R}$ if and only if $\mathfrak{A}_k \in \mathrm{GL}_k(K)$ for $k = 1, \ldots, n$. (**Remark :** Therefore we have a criterion : In the case of invertible matrices in Theorem 8.C.9 and Test-Exercise T8.41-(b), when exactly we do not need the permutation matrix. In particular, in the case of a positive or negative definite real-symmetric or complex-hermitian matrices $\mathfrak{A}$, there exist $\mathfrak{L}$ and $\mathfrak{R}$. Moreover, if we choose $\mathfrak{L}$ normalized, then $\mathfrak{L}$ and $\mathfrak{R}$ are uniquely determined.)

T8.43 Compute the product representation corresponding to Theorem 8.C.8, Theorem 8.C.9 and Test-Exercise T8.41 for the matrices

$$\mathfrak{A} := \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ -2 & -1 & 2 \end{pmatrix} \quad \text{respectively} \quad \mathfrak{A} := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and therefore determine $\mathfrak{A}^{-1}$ in $\mathrm{GL}_3(\mathbb{R})$.

T8.44 Let

$$\mathfrak{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \in \mathrm{M}_5(\mathbb{R}).$$

Compute a normalized lower triangular matrix $\mathfrak{L}$ and an upper triangular matrix $\mathfrak{R}$ such that $\mathfrak{A} = \mathfrak{L}\mathfrak{R}$.

T8.45 Suppose that the well-known t r i - d i a g o n a l m a t r i x

$$\mathfrak{A} = \begin{pmatrix} a_1 & c_1 & 0 & \cdots & 0 & 0 \\ b_2 & a_2 & c_2 & \cdots & 0 & 0 \\ 0 & b_3 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & c_{n-1} \\ 0 & 0 & 0 & \cdots & b_n & a_n \end{pmatrix} \in \mathrm{M}_n(K)$$

satisfy the equivalent conditions of the Test-Exercise T8.42, i. e. all principal minors $\mathrm{Det}\mathfrak{A}_k \neq 0$ for all $k = 1, \ldots, n$. Show that (by induction on n), there exists a normalised lower triangular matrix $\mathfrak{L}$ of the form

$$\mathfrak{L} = \mathfrak{B}_{21}(\beta_2)\,\mathfrak{B}_{32}(\beta_3)\cdots\mathfrak{B}_{n,n-1}(\beta_n) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \beta_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \beta_n & 1 \end{pmatrix}$$

and an upper triangular matrix $\mathfrak{R}$ of the form

$$\mathfrak{R} = \begin{pmatrix} \alpha_1 & c_1 & \cdots & 0 & 0 \\ 0 & \alpha_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & \alpha_n \end{pmatrix}$$

in $\mathrm{M}_n(K)$ such that $\mathfrak{A} = \mathfrak{L}\mathfrak{R}$.