E0 219 Linear Algebra and Applications / August-December 2011

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

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1-st Midterm : Saturday, September 17, 2011; 15:00 -17:00 **2-nd Midterm :** Saturday, October 22, 2011; 10:30 -12:30
Final Examination : December ??, 2011, 10:00 -13:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

9. Determinants

Submit a solution of the $*$-Exercise ONLY

Due Date : Monday, 14-11-2011 (Before the Class)

- **Solution of the $**$-Exercise (Exercise 9.7) carries 10 Bonus Points.**
- **Solution of the $***$-Exercise (Exercise 9.2)) carries 20 Bonus Points!**

9.1 (Boss-Puzzle) Let $r,s \in \mathbb{N}^*$, $r,s \geq 2$. In an right side box there are $rs-1$ numbers $1,2,\ldots,rs-1$ are arranged in a $r \times s$-rectangle (as shown in the left-rectangle which is made up of equal rs sliding square-blocks) by the permutation

$$\nu = \begin{pmatrix} 1 & 2 & 3 & \cdots & rs-2 & rs-1 \\ \nu_1 & \nu_2 & \nu_3 & \cdots & \nu_{rs-2} & \nu_{rs-1} \end{pmatrix} \in \mathfrak{S}_{rs-1}$$

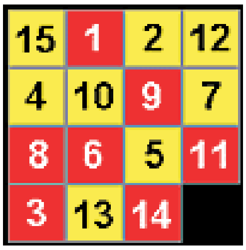
ν_1	$\cdots$	ν_{s-1}	ν_s
ν_{s+1}	$\cdots$	ν_{2s-1}	ν_{2s}
$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\nu_{(r-1)s+1}$	$\cdots$	ν_{rs-1}	#

1	$\cdots$	$s-1$	s
$s+1$	$\cdots$	$2s-1$	$2s$
$\vdots$	$\cdots$	$\vdots$	$\vdots$
$(r-1)(s-1)+1$	$\cdots$	$rs-1$	#

The lower-right corner square-block marked with # is kept free. The goal is to reposition the square-blocks by sliding the square-blocks (one at a time) into the standard-configuration (shown in left-hand table). Show that this possible if and only if the permutation $\nu \in \mathfrak{S}_{rs-1}$ is even.
(**Remark:** The special case $r=4$ and $s=4$ is the (original) 15-puzzle [1]:

[1] The 15-puzzle (also called Gem Puzzle, Boss Puzzle, Game of Fifteen, Mystic Square and many others) was "invented" by Noyes Palmer Chapman, a postmaster in Canastota, New York as early as 1874. The game became a craze in the U. S. in February 1880, Canada in March, Europe in April, but that craze had pretty much dissipated by July.

Samuel Loyd (1841-1911) an American chess player-composer, puzzle author, and recreational mathematician, claimed from 1891 until his death in 1911 that he invented the 15-puzzle. This is false – Loyd had nothing to do with the invention or popularity of the puzzle. Later interest was fuelled by Loyd offering a \$1,000 prize for anyone who could provide a solution for achieving a particular combination specified by Loyd, namely reversing the 14 and 15, i. e. $\sigma = \langle 14, 15 \rangle$. This was impossible, as had been shown over a decade earlier by Johnson and Story (1879), (see: [Johnson, W. W.; Story, W. E.: Notes on the 15-Puzzle, *American Journal of Mathematics*, **2** (4), (1879), 397-404]) as it required an even permutation. Robert James "Bobby" Fischer (1943-2008) an American chess Grandmaster and the 11-th World Chess Champion, was an expert at solving the 15-Puzzle and had demonstrated on Nov. 8, 1972 a solution within 25 seconds. Today the puzzle appears on some computer screen savers and a version is distributed with every Macintosh computer. For larger versions of the n-puzzle, finding a solution is easy, but the problem of finding the shortest solution is *NP-hard* (??).



This puzzle has inspired a sizable number of articles and references in the mathematical literature. Most references explain the impossibility of obtaining odd permutations, but the result that every even permutation is indeed possible is proved by few authors and a number of them give unnecessarily complicated explanations. Indeed, Herstein and Kaplansky in (see: [Herstein, I. N. and Kaplansky, K.: *Matters Mathematical*, Chelsea, New York, 1978, 114-115]) write that "no really easy proof seems to be known". – **Hint:** A simple move interchanges the blank-square # with adjacent to it; for example, there are two beginning simple moves, namely, either interchange # and v_{rs-1} or interchange # and $v_{(r-1)s}$. To analyze the game, note that each simple move is a special kind of transposition, namely, one that moves #. Moreover, performing a simple move corresponding to a special transposition τ from a position corresponding to the permutation σ) yields a new position (corresponding to the permutation $\tau\sigma$). For example, if v is the position above and $\tau = \langle \#, v_{rs-1}\rangle$, then $\tau v(\#) = \tau(\#) = v_{rs-1}$, $\tau v(rs-1) = \tau(v_{rs-1}) = \#$ and $\tau v(i) = i$ for all other i. Therefore to come to the standard position, one needs special transpositions $\tau_1, \tau_2, \ldots, \tau_m$ such that $\tau_m \cdots \tau_2\tau_1 v = \mathrm{id}$. Each simple move takes # up, down , left or right. Therefore the total number m of moves is $u+d+\ell+r$, where u, d, ℓ, r are the numbers of up, down, left, right moves, respectively. If # is to return at the position where it was, then $u = d$ and $\ell = r$. Therefore the total number of moves must be $m = 2u+2r$ even. The permutation $v \in \mathfrak{S}_{16}$ corresponding to the configuration in the above picture $v = \langle 1,15,14,13,3,2\rangle\langle 4,12,11,5\rangle\langle 6,10\rangle\langle 7,9,8\rangle$ is an odd permutation and hence it is not possible to bring it to the standard configuration. For the converse, use Test-Exercise T9.11-(f) to reduce the problem to the cases $s = 2$, $r = 2$ or 3. – The permutations for which this is possible form a subgroup of $\mathfrak{S}_n$, in fact it is the alternating group $\mathfrak{A}_n$ on n symbols.)

*****9.2** For $1 \le i < n$, let m_i be the number of *inversions*[2] (i,j), $i<j\le n$, in the permutation $\sigma \in \mathfrak{S}_n$ and let $\sigma_i := \langle i+m_i, i+m_i-1\rangle \cdots \langle i+1, i\rangle$. Show that $\sigma = \sigma_1 \cdots \sigma_{n-1}$. (**Remark:** This proves 9.A.18 again and one can recover the permutation σ from its inversions. More precisely: The permutation σ is uniquely determined by the $(n-1)$-tuple $(m_1, \ldots, m_{n-1})$ with $0 \le m_i \le n-i$ and every such tuple uniquely determine a permutation $\sigma \in \mathfrak{S}_n$. This encoding of the elements of $\mathfrak{S}_n$ is frequently used. – One can also examine the analogous problem with the numbers m'_i of the inversions (j,i), $j<i$, $i = 2, \ldots, n$.)

9.3 Let V and W be vector spaces over a field K and let I be a finite indexed set with n elements.

(a) Suppose that in K the element $n! = n!\cdot 1_K$ is non-zero, i.e. $\mathrm{Char}\,K = 0$ or $\mathrm{Char}\,K > n$. Then the maps $f \mapsto \frac{1}{n!}Af$ and $f \mapsto \frac{1}{n!}Sf$ are projections of the K-vector space of the multi-linear maps $V^I \to W$ onto the subspace of the alternating respectively, the symmetric I-linear maps.

(b) Suppose that $\mathrm{Char}\,K \neq 2$. The space of the bilinear maps $V \times V \to W$ is the direct sum of the subspace of the alternating (i. e. skew-symmetric) and the subspace the symmetric bilinear maps. The corresponding projections are $\frac{1}{2}A$ resp. $\frac{1}{2}S$. (**Remark :** A bilinear map $f: V\times V \to W$ can be decomposed into its skew-symmetric part $\frac{1}{2}Af$ and its symmetric part $\frac{1}{2}Sf$.)

9.4 **(a)** (Cramer's[3] Formula) Suppose that V is a n-dimensional vector space over a field K. For every determinant function $\Delta\colon V^n \to K$ and for arbitrary $x_0, \ldots, x_n \in V$, prove that

$$\sum_{i=0}^{n} (-1)^i \Delta(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots x_n)\, x_i = 0\,.$$

[2] Inversions of a permutation $\sigma \in \mathfrak{S}_n$ are the pairs (i,j) such that $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$.

[3] Gabriel Cramer (1704-1752) was a Swiss mathematican who worked on analysis and determinants. He is best known for his formula for solving simultaneous equations.

(**Hint :** The left-hand side is by Test-Exercise T9.23 (with $g = \mathrm{id}_V$) is an alternating multi-linear map $V^{n+1} \to V$, and hence vanish by Corollary 9.B.6, since $\mathrm{Dim}\, V = n$.)

(b) Let $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$ be an $m \times n$-matrix over a field K and let $\mathrm{Det}\,\mathfrak{A}_{I,J}$ be a non-zero r-minor of $\mathfrak{A}$. Show that $\mathrm{Rank}\,\mathfrak{A} = r$ if and only if every $(r+1)$-minor $\mathrm{Det}\,\mathfrak{A}_{I',J'} = 0$ with $I \subset I'$ and $J \subset J'$.

9.5 **(a)** (Vandermonde's determinant) For elements $a_0, \ldots, a_n \in K$, show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0^n & a_1^n & a_2^n & \cdots & a_n^n \end{vmatrix} = \prod\nolimits_{0 \le i < j \le n} (a_j - a_i)\,. \quad (\textbf{Hint :} \text{ Induction on } n. \text{ -- See Exercise 8.3.})$$

(b) (Cauchy's Double-alternant) Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in K$ with $a_i + b_j \neq 0$ for all $i, j = 1, \ldots, n$. Show that

$$\mathrm{Det}\left(\left(\frac{1}{a_i + b_j}\right)_{1 \le i,j \le n}\right) = \frac{\prod_{1 \le i < j \le n}(a_j - a_i)\, \prod_{1 \le i < j \le n}(b_j - b_i)}{\prod_{i,j=1}^{n}(a_i + b_j)}\,.$$

(**Hint :** Induction on n. – See also Exercise 8.4.)

*__9.6__ **(a)** Show that

$$\begin{vmatrix} 1^n & 2^n & 3^n & \cdots & (n+1)^n \\ 2^n & 3^n & 4^n & \cdots & (n+2)^n \\ 3^n & 4^n & 5^n & \cdots & (n+3)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n+1)^n & (n+2)^n & (n+3)^n & \cdots & (2n+1)^n \end{vmatrix} = (-1)^{\binom{n+1}{2}}\, (n!)^{n+1}\,.$$

(**Hint :** Since $(i+j-1)^n = \sum_{k=1}^{n+1} \binom{n}{k-1} i^{k-1} (j-1)^{n+1-k}$, the above matrix is the product of two matrices and their determinants can be computed by using the Vandermonde's determinant, see Exercise 9.5-(a).)

(b) Compute the determinant of the $n \times n$ matrix over a field K:

$$\begin{vmatrix} 1 + a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & 1 + a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & 1 + a_n b_n \end{vmatrix}\,.$$

(**Hint:** If all $a_i = 0$, then it is the identity matrix and hence its determinant is 1. Otherwise, we may assume that $a_n \neq 0$. For $i = 1, \ldots, n-1$, replace i-th row by adding $-a_i a_n^{-1}$-times the n-th row to it and then replace the last row by by adding the $-a_n b_i$-times the i-th row, we get an upper triangular matrix:

$$\begin{vmatrix} 1 + a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & 1 + a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & 1 + a_n b_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & -a_1 a_n^{-1} \\ 0 & 1 & \cdots & -a_2 a_n^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & 1 + a_n b_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & -a_1 a_n^{-1} \\ 0 & 1 & \cdots & -a_2 a_n^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \sum_{i=1}^{n} a_i b_i \end{vmatrix} = 1 + \sum_{i=1}^{n} a_i b_i\,.)$$

(c) Solve the following system of linear equations by using Cramer's rule:

$$\begin{array}{rcl} x_2 + x_3 + \cdots + x_{n-1} + x_n &=& 1 \\ x_1 + x_3 + \cdots + x_{n-1} + x_n &=& 1 \\ x_1 + x_2 + \cdots + x_{n-1} + x_n &=& 1 \\ \cdots\cdots \quad \cdots \quad \cdots\cdots \quad & & \cdots \\ x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n &=& 1 \end{array}$$

**9.7 Let $\mathfrak{A}=(a_{ij})\in \mathrm{M}_n(\mathbb{R})$ be $n\times n$-matrix over real numbers .

(a) Suppose that for each $i=1,\ldots,n$, $|a_{ii}|>\sum_{j=1,j\neq i}^{n}|a_{ij}|$. Then $a_{11}\cdots a_{nn}\operatorname{Det}\mathfrak{A}>0$. (**Hint:** By Exercise 4.2 $\operatorname{Det}\mathfrak{A}\neq 0$ for such a matrix. Therefore the continuous polynomial function

$$f(t):=\begin{vmatrix} a_{11} & ta_{12} & ta_{13} & \cdots & ta_{1n}\\ ta_{21} & a_{22} & ta_{23} & \cdots & ta_{2n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ ta_{n-1,1} & ta_{n-1,2} & ta_{n-1,3} & \cdots & ta_{n-1,n}\\ ta_{n1} & ta_{n2} & ta_{n3} & \cdots & a_{nn}\end{vmatrix}$$

has no zero in the interval $[0,1]$ and so the values $f(0)$ and $f(1)$ have the same sign by the *Intermediate Value Theorem* [4]. – **Remarks:** Two important special cases are: (1) (H a d a m a r d [5]) Let $(a_{ij})\in\mathrm{M}_n(\mathbb{C})$. For every i, suppose that in the i-th row there is at most one element $a_{i,j(i)}\neq 0$ with $j(i)\neq i$ and for this element $|a_{i,j(i)}|<|a_{ii}$. Then the matrix (a_{ij}) is invertible. (2) (M i n k o w s k i [6]) Let $(a_{ij})\in\mathrm{M}_n(\mathbb{R})$ with $a_{ij}\leq 0$ for every $i\neq j$. For every i, suppose that $\sum_{j=1}^{n}a_{ij}>0$. Then the matrix (a_{ij}) is invertible.)

(b) Suppose that $n\in\mathbb{N}$ is odd. Then show that there exists a real $t\in\mathbb{R}$ such that

$$\operatorname{Det}\begin{pmatrix} a_{11}+t & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22}+t & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}+t\end{pmatrix}=0\,.$$

(**Hint:** The determinant is a polynomial of odd degree n in t and hence it has (by intermediate value theorem, see the Footnote 4) a zero in $\mathbb{R}$.)

9.8 Let V be a finite dimensional K-vector space. Compute the determinant of the K-linear map $f:V\to V$ in the following cases: **(a)** f is the homothecy $a\,\mathrm{id}_V$. **(b)** f is a projection. **(c)** f is an involution (see Test-Ex. T8.9-(b)). **(d)** f is a transvection or a dilatation (see Test-Ex. T8.9-(c), (d)).

9.9 **(a)** Let V be an oriented n-dimensional $\mathbb{R}$-vector space and let $\sigma\in\mathfrak{S}_n$ be a permutation. Suppose that the orientation of V is represented by the $v_1,\ldots,v_n$. Show that $v_{\sigma(1)},\ldots,v_{\sigma(n)}$ represent the orientation of V if and only if σ is an even permutation. Further, show that the basis $v_n,\ldots,v_1$ represent the orientation of V if and only if $n\equiv 0$ or $n\equiv 1$ modulo 4.

(b) The bases $\mathfrak{E}_{11},\ldots,\mathfrak{E}_{1n},\ldots,\mathfrak{E}_{m1},\ldots,\mathfrak{E}_{mn}$ and $\mathfrak{E}_{11},\ldots,\mathfrak{E}_{m1},\ldots,\mathfrak{E}_{1n},\ldots\mathfrak{E}_{mn}$ represent the same orientation of $\mathrm{M}_{m,n}(\mathbb{R})$ if and only if $\binom{m}{2}\binom{n}{2}$ is even.

9.10 **(a)** Show that the volume of the ellipsoid

$$\Big\{(x_1,\ldots,x_n)\in\mathbb{R}^n \;\Big|\; \frac{x_1^2}{a_1^2}+\cdots+\frac{x_n^2}{a_n^2}\leq 1\Big\}\subseteq\mathbb{R}^n\,,$$

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[4] **Intermediate Value Theorem** *Every continuous function $f:[a,b]\to\mathbb{R}$ on an interval $[a,b]$ attains every value in between $f(a)$ and $f(b)$,* i. e. *for every $c\in\mathbb{R}$ in between $f(a)$ and $f(b)$ there exists $x_0\in[a,b]$ such that $f(x_0)=c$.*
– For c = 0 above, the statement is also known as Bolzano's theorem. This theorem was first proved by B e r n a r d B o l z a n o (1781-1848) (a mathematicain from Prague, Bohemia, Austrian Habsburg domain, now Czech Republic, who successfully freed calculus from the concept of the infinitesimal. He also gave examples of 1-1 correspondences between the elements of an infinite set and the elements of a proper subset.) in 1817. A French mathematician A u g u s t i n L o u i s C a u c h y (1789-1857) (Cauchy pioneered the study of analysis, both real and complex, and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics) provided a proof in 1821. Both were inspired by the goal of formalizing the analysis of functions and the work of Lagrange. The insight of Bolzano and Cauchy was to define a general notion of continuity (in terms of infinitesimals in Cauchy's case, and using real inequalities in Bolzano's case), and to provide a proof based on such definitions.

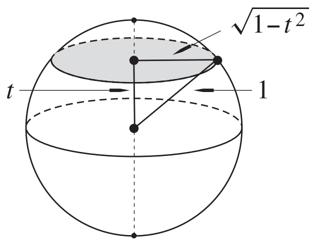
[5] J a c q u e s H a d a m a r d (1865-1963) was a French mathematician whose most important result is the prime number theorem which he proved in 1896. This states that the number of primes $<n$ tends to infinity as fast as $n/\ln n$.

[6] H e r m a n n M i n k o w s k i (1864-1909) was a German mathematician who developed a new view of space and time and laid the mathematical foundation of the theory of relativity.

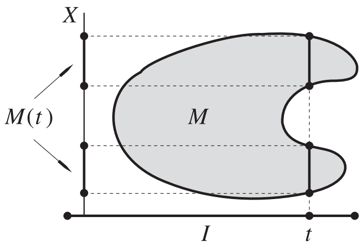
$a_1,\ldots,a_n\in\mathbb{R}_+^\times$, is $\omega_n\, a_1\cdots a_n$, where ω_n is the volume of the unit-ball

$$\overline{\mathrm{B}}^{\,n-1} := \left\{(x_1,\ldots,x_n)\in\mathbb{R}^n \;\middle|\; x_1^2+\cdots+x_n^2\le 1\right\}.$$

(**Remark:** The volume of the unit-sphere is $\omega_n=\pi^{n/2}/(n/2)!$. Note that, for $z\in\mathbb{C}$ and $m\in\mathbb{N}$, $\binom{z}{m} := \frac{[z]_m}{m!} := \frac{z(z-1)\cdots(z-m+1)}{1\cdot 2\cdots(m-1)\cdot m}$. – To compute the volume [7] of the unit-ball $\overline{B}^{\,n}:=\overline{\mathrm{B}}(0;1)=\{x\in\mathbb{R}^n \mid \|x\|\le 1\}$ in $\mathbb{R}^n$, where $\|-\|$ denote the standard Euclidean norm. We put $\omega_n:=\lambda^n(\overline{B}^{\,n})$. The volume of a ball with radius r is then $\omega_n r^n$. (Why ? see also Exercise ?.??) It is easy to check that $\omega_0=1,\ \omega_1=2,\ \omega_2=\pi$ and the equality of A r c h i m e d e s: $\omega_3=\frac{4}{3}\pi$, since the surface-area $\lambda^2\big((\{t\}\times\mathbb{R}^2)\cap\overline{B}^{\,3}\big)=\pi(1-t^2)$, $-1\le t\le 1$, is a polynomial of degree 2 (≤ 3) in t.



– First, we would like to generalise the Fundamental Theorem of Differential–and Integral Calculus as follows: Let $I\subseteq\mathbb{R}$ be an interval, $X=(X,\mathscr{A},\mu)$ be a σ-finite measure space and $M\subseteq I\times X$ be a measurable set (in the product measure space $(I\times X,\mathscr{B}^1\otimes\mathscr{A},\lambda^1\otimes\mu)$). For every $t\in I$, we consider the set $M(t):=\{x\in X \mid (t,x)\in M\}\subseteq X$.



For every $t\in I$, the set $M(t)$ is a measurable set in X. Suppose that the following conditions are satisfied:

1) For every $t\in I$, we have $\mu\big(M(t)\big)<\infty$.

2) For every $t_0\in I$ and every $\varepsilon>0$, there exists a $\delta>0$ and measurable sets $M',M''\subseteq X$ such that $M'\subseteq M''$, $\mu(M'')\le\mu(M')+\varepsilon$, and $M'\subseteq M(t)\subseteq M''$ for all $t\in I$, $|t-t_0|\le\delta$.

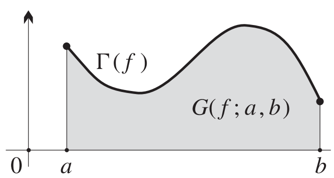
From (2) it is immediate that the function $t\mapsto\mu\big(M(t)\big)$ on I is continuous. Moreover:

Theorem *Let $a,b\in\overline{\mathbb{R}}$, $a\le b$, be the end-points of the interval I with $M\subseteq I\times X$. Then* $(\lambda^1\otimes\mu)(M)=\int_a^b\mu\big(M(t)\big)\,dt$.

If the interval $I=[a,b]$ in the above theorem is compact of the length $h:=b-a$ and the measures $\mu\big(M(a)\big)$,

[7] In general it is difficult to compute the (volume=) Borel-Lebesgue measure $\lambda^n(M)$ of an arbitrary Borel-set $M\subseteq\mathbb{R}^n$. For subsets in $\mathbb{R}^2$, we have used the *Fundamental Theorem of Differential–and Integral Calculus*:

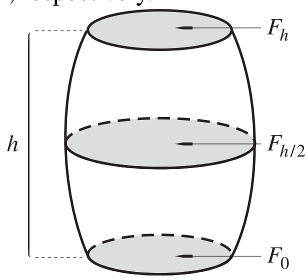
Theorem (F u n d a m e n t a l T h e o r e m o f D i f f e r e n t i a l – a n d I n t e g r a l C a l c u l u s) *Let $f:[a,b]\to\mathbb{R}$, $a\le b$, be a continuous function with $f\ge 0$. Then the integral $\int_a^b f(t)\,dt$ is the area of the compact set $G(f;a,b):=\{(x,y)\mid a\le x\le b,\ 0\le y\le f(x)\}$.*



$\mu\big(M(\frac{1}{2}(a+b))\big)$ and $\mu\big(M(b)\big)$ are denoted by $F_0, F_{h/2}, F_h$, respectively, then we get the Simpson-Rule[8] for the approximation of the volume

$$V=(\lambda^1\otimes\mu)(M)\approx\frac{h}{6}\,(F_0+4F_{h/2}+F_h)\,,$$

where this approximation is exact if $t\mapsto\mu\big(M(t)\big)$ is a polynomial function of degree ≤ 3. This rule is also known as the Kepler's barrel-rule[9]: It approximate the volume V of a barrel using the height h, areas $F_0, F_{h/2}$ and F_h of the base, middle-place and top of the barrel, respectively.



– This is a very special case of a more general theorem proved by an Italian mathematician (Bonaventura Francesco Cavalieri (1598-1647) who developed a method of indivisibles which became a factor in the development of the integral calculus.)

Now, by the above Theorem, for $n\in\mathbb{N}$, we have the recursion-formula:

$$\omega_{n+1}=\int_{-1}^{1}\lambda^n\big(\overline{\mathrm{B}}^n(0;\sqrt{1-t^2})\big)\,dt=\omega_n\int_{-1}^{1}\big(\sqrt{1-t^2}\big)^n\,dt\,,$$

and so $\omega_n=2^n c_1\cdots c_n$, where

$$c_k:=\frac{1}{2}\int_{-1}^{1}\big(\sqrt{1-t^2}\big)^{k-1}\,dt=\int_0^{\pi/2}\sin^k t\,dt=\begin{cases}\dfrac{\pi}{2}\cdot\dfrac{1}{4^m}\dbinom{2m}{m}, & \text{if } k=2m \text{ is even,}\\[2ex] \dfrac{4^m}{2m+1}\Big/\dbinom{2m}{m}, & \text{if } k=2m+1 \text{ is odd}\end{cases}$$

It follows that

$$\omega_n=\begin{cases}2^{m+1}\pi^m/1\cdot3\cdots(2m+1)\,, & \text{if } n=2m+1 \text{ is odd,}\\ \pi^m/m!\,, & \text{if } n=2m \text{ is even.}\end{cases}$$

By using the Γ-*function*[10] this result may be expressed as (in a more appreciated form):

$$\omega_n=\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}=\frac{\pi^{n/2}}{(n/2)!}\sim\frac{1}{\sqrt{\pi n}}\left(\frac{2\pi e}{n}\right)^{n/2}.$$

(by using the well-known Wallis[11] product representation: $\sqrt{\pi}=\lim\limits_{m\to\infty}\dfrac{2\cdot4\cdots(2m)}{1\cdot3\cdots(2m-1)}$ and the *Striling's formula* for the last asymptotic representation for $n\to\infty$. For which $x_0\in\mathbb{R}_+$, the function $x\mapsto\pi^x/x!$ has maximum at x_0?)

(b) Sketch the following set $M:=H_1\cap H_2\cap H_3$ in $\mathbb{R}^2$ with $H_i:=\big\{(x,y)\in\mathbb{R}^2\;\big|\;f_i(x,y)\geq 0\big\}$, $i=1,2,3$, and $f_1(x,y):=x+3y+1$, $f_2(x,y):=-5x+y+1$, $f_3(x,y):=x-y+3$ and calculate its surface-area.

On the other side one can see auxiliary results and (simple) Test-Exercises.

[8] Thomas Simpson (1710-1761) was an English mathematician who is best remembered for his work on interpolation and numerical methods of integration.

[9] Johannes Kepler (1571-1630) was a German mathematician and astronomer who discovered that the Earth and planets travel about the sun in elliptical orbits. He gave three fundamental laws of planetary motion. He also did important work in optics and geometry.

[10] The function $\Gamma(x):=\int_0^\infty t^{x-1}e^{-t}dt$, $\mathrm{Re}\,x>0$, is called the Γ(gamma)-function. The following computational rules are important: (1) $\Gamma(x+1)x\Gamma(x)$. (2) $\Gamma(n+1)=n!$ for all $n\in\mathbb{N}$. (3) $\Gamma(\frac{1}{2}(2n+1))=1\cdot3\cdots(2n-1)\sqrt{\pi}/2^n$ for all $n\in\mathbb{N}$.

[11] John Wallis (1616-1703) was an English mathematician who built on Cavalieri's method of indivisibles to devise a method of interpolation. Using Kepler's concept of continuity he discovered methods to evaluate integrals.

Auxiliary Results/Test-Exercises

To understand and appreciate the Test-Exercises which are marked with the symbol † one may possibly require more mathematical maturity than one has! These are steps towards applications to various other branches of mathematics, especially to Analysis, Number Theory, Graph Theory, Group Theory and Affine and Projective Geometry.

T9.1 **(a)** Give an element of biggest possible order in the group $\mathfrak{S}_5$.

(b) For $n \geq 4$, the group $\mathfrak{A}_n$ is not abelian.

T9.2 For the following permutations compute the number of inversions and the *signum*.[12]

(a) The permutation $i \mapsto n-i+1$ in $\mathfrak{S}_n$.

(b) $\begin{pmatrix} 1 & 2 & \ldots & n & n+1 & \ldots & 2n \\ 1 & 3 & \ldots & 2n-1 & 2 & \ldots & 2n \end{pmatrix} \in \mathfrak{S}_{2n}$.

(c) $\begin{pmatrix} 1 & 2 & \ldots & n & n+1 & \ldots & 2n \\ 2 & 4 & \ldots & 2n & 1 & \ldots & 2n-1 \end{pmatrix} \in \mathfrak{S}_{2n}$.

(d) $\begin{pmatrix} 1 & \ldots & n-r+1 & n-r+2 & \ldots & n \\ r & \ldots & n & 1 & \ldots & r-1 \end{pmatrix} \in \mathfrak{S}_n,\ 1 \leq r \leq n$. (**Ans :** $(-1)^{(r-1)(n+1)}$.)

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & 2n \\ 1 & 2n & 3 & 2(n-1) & 5 & 2(n-2) & \ldots & 2 \end{pmatrix} \in \mathfrak{S}_{2n}$.

T9.3 For a subset $J \subseteq \{1,\ldots,n\}$ with $J = \{j_1,\ldots,j_m\}$, $j_1 < \cdots < j_m$, let σ_J be the permutation

$$\sigma_J = \begin{pmatrix} 1 & \ldots & m & m+1 & \ldots & n \\ j_1 & \ldots & j_m & i_1 & \ldots & i_{n-m} \end{pmatrix} \in \mathfrak{S}_n\,,$$

where the numbers $i_1 < \cdots < i_{n-m}$ are the elements of the complement of J in $\{1,\ldots,n\}$. (**Hint :** The number of inversions of σ_J is $F(\sigma_J) = \left(\sum_{k=1}^m j_k\right) - \binom{m+1}{2}$ and hence $\operatorname{Sign}(\sigma_J) = (-1)^{F(\sigma_J)}$.)

T9.4 Let σ respectively τ be permutations of the finite sets I respectively J. Compute the sign of the permutation $\sigma \times \tau : (i, j) \mapsto (\sigma i, \tau j)$ of $I \times J$ (in terms of $\operatorname{Sign}\sigma$, $\operatorname{Sign}\tau$ and $m := |I|$, $n := |J|$).

T9.5 A subgroup of the permutation group $\mathfrak{S}_n$, $n \in \mathbb{N}^+$, which contain an odd permutation contains equal number of even and odd permutations.

T9.6 **(a)** A permutation $\sigma \in \mathfrak{S}_n$, $n \in \mathbb{N}^+$, which is of odd order is an even permutation.

(b) The square σ^2 of a permutation $\sigma \in \mathfrak{S}_n$, $n \in \mathbb{N}^+$, is an even permutation.

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[12] S i g n u m is the Latin word for "mark" or "token", of course, it has become the word s i g n a t u r e or just s i g n. Another notation for the sign of a permutation is given by the more general L e v i - C i v i t a s y m b o l ε_σ, which is defined for all maps from $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$, and has value zero for non-bijective maps, in fact: $\varepsilon_\sigma = \prod_{i=1}^{n-1} \left(\frac{1}{i!} \prod_{j=i+1}^{n} (\sigma(j) - \sigma(i)) \right)$. The Levi-Civita symbol, also called the a n t i s y m m e t r i c s y m b o l, or a l t e r n a t i n g s y m b o l, is a mathematical symbol used in particular in *tensor calculus*. It is named after the Italian mathematician and physicist T u l l i o L e v i - C i v i t a (1873-1941). In 1900 he and G r e g o r i o R i c c i - C u r b a s t r o (1853-1925) published the theory of tensors in "Méthodes de calcul différentiel absolu et leurs applications", which A l b e r t E i n s t e i n (1879-1955) (Einstein contributed more than any other scientist to the modern vision of physical reality. His special and general theories of relativity are still regarded as the most satisfactory model of the large-scale universe that we have) used as a resource to master the tensor calculus, a critical tool in Einstein's development of the theory of general relativity. In one of the letters, regarding Levi-Civita's work, Einstein wrote "I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot". In 1933 Levi-Civita contributed to Paul Dirac's equations in quantum mechanics as well.

T9.7 Let $\sigma = \langle i_0, \ldots, i_{k-1}\rangle$ be a cycle of length $k \geq 2$. What is the inverse of σ ? For which $m \in \mathbb{Z}$, σ^m is a cycle of length k?

T9.8 Let $\sigma \in \mathfrak{S}_n$ and $m \in \mathbb{Z}$. Every orbit of σ of length k decomposes into $\gcd(k,m)$ orbits of the length $k/\gcd(k,m)$ of σ^m.

T9.9 Let I be a finite set. The inverse σ^{-1} of a permutation $\sigma \in \mathfrak{S}(I)$ has the same orbits and same sign as those of σ.

T9.10 Let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the canonical prime factorisation of $m \in \mathbb{N}^*$. Then the permutation group $\mathfrak{S}_n$ contain an element of order m if and only if $n \geq p_1^{\alpha_1} + \cdots + p_r^{\alpha_r}$. For which $n \in \mathbb{N}$ there exists an element of order 3000 (respectively 3001) in the group $\mathfrak{S}_n$?

†**T9.11** Let T be a set of transpositions in the group $\mathfrak{S}_n$, $n \geq 1$. We associate the *graph* [13] Γ_T to T as follows: the vertices of Γ_T are the numbers $1, \ldots, n$ and two vertices i and j with $i \neq j$ are joined by a edge if and only if the transposition $\langle i, j\rangle = \langle j, i\rangle$ belong to T. Let $\Gamma_1, \ldots, \Gamma_r$ be the connected components of Γ_T.

(a) The transpositions in T *generate the group*[14] $\mathfrak{S}_n$ if and only if Γ_T is connected, i.e. if any two vertices of Γ_T can be joined by the sequence of edges in Γ_T. The subgroup of $\mathfrak{S}_n$ generated by T is the product $\mathfrak{S}(\Gamma_1) \times \cdots \times \mathfrak{S}(\Gamma_r) \subseteq \mathfrak{S}_n$.

(b) If T is a generating system for the group $\mathfrak{S}_n$, then T has at least $n-1$ elements. (**Hint :** Let $\tau_1, \ldots, \tau_m$ be the elements of T (may be with repetitions) with $\tau_1 \cdots \tau_m = \mathrm{id}$. Then m is even and $m \geq 2\sum_{\rho=1}^{r}(|\Gamma_\rho| - 1)$.)

(c) Every generating system of $\mathfrak{S}_n$ consisting of transpositions contain a (minimal) generating system of $\mathfrak{S}_n$ with $n-1$ elements. (**Remarks :** The graphs corresponding to such a minimal generating systems are called trees. Every connected graph has a generating system which is a tree. See also remarks in Subsection 6.D. –There are exactly n^{n-2} generating systems consisting $n-1$ transpositions (Cayley[15]). For this prove somewhat general: For $1 \leq k \leq n$, let $f_{n,k}$ denote the number of forests with the vertex set $\{1, \ldots, n\}$ and exactly k *marked* trees (so-called root-trees), then $f_{n,n} = 1$, $(n-k+1)f_{n,k-1} = n(k-1)f_{n,k}$ (by "grafting" one can get from a forest with $k \geq 2$ root-trees $n(k-1)$ forest with $k-1$ root-trees and by

[13] **Simplicial Complexes and Graphs.** A simplicial complex $\mathscr{K}$ is a set $\mathbf{V}(\mathscr{K})$ called the vertex set (of $\mathscr{K}$) and a family of subsets of $\mathbf{V}(\mathscr{K})$, called simplexes (in $\mathscr{K}$) such that (i) for each $v \in \mathbf{V}(\mathscr{K})$, the singleton set $\{v\}$ is a simplex in K. and (ii) if $\mathbf{s}$ is a simplex in $\mathscr{K}$ then so is every subset of $\mathbf{s}$.

A simplex $\mathbf{s}$ in $\mathscr{K}$ is called a q-simplex if $\mathrm{card}(\mathbf{s}) = q+1$ and say that $\mathbf{s}$ has dimension q. For a simplicial complex $\mathscr{K}$, we put $\dim(\mathscr{K}) := \sup\{q \mid$ there exists a q-simplex in $\mathscr{K}\}$ and is called the dimension of $\mathscr{K}$. A simplicial complex of dimension ≤ 1 is called a graph.

An edge in $\mathscr{K}$ is an ordered pair (v_0, v_1) of vertices such that $\{v_0, v_1\}$ is a simplex in $\mathscr{K}$. If $\mathbf{e} = (v_0, v_1)$ is an edge in $\mathscr{K}$, then we put $v_0 = \alpha(\mathbf{e})$ and $v_1 = \varepsilon(\mathbf{e})$) and are called the initial and end points of $\mathbf{e}$, respectively.

A path γ in $\mathscr{K}$ of length n is a sequence $\mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$ of edges in K with $\varepsilon(\mathbf{e}_i) = \alpha(\mathbf{e}_{i+1})$ for every $1 \leq i \leq n-1$. For a path $\gamma = \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$, we put $\alpha(\gamma) = \alpha(\mathbf{e}_1)$ and $\varepsilon(\gamma) := \varepsilon(\mathbf{e}_n)$ and are called the initial and end points of γ.

A simplicial complex $\mathscr{K}$ is called connected if for every pair (v_0, v_1) of vertices in $\mathscr{K}$ there exists a path α in $\mathscr{K}$ such that $\mathrm{orig}(\alpha) = v_0$ and $\mathrm{end}(\alpha) = v_1$.

[14] The smallest subgroup $\mathrm{H}(a_i \mid i \in I)$ of a group G containing a family a_i, $i \in I$, of elements in G, is called the subgroup generated by the family $a_i, i \in I$ (it is the intersection of the subgroups of G containing all a_i, $\in I$) and the family a_i, $i \in I$, is called a generating system for the subgroup $\mathrm{H}(a_i \mid i \in I)$. A family a_i, $i \in I$, is called a generating system for the group G if $G = \mathrm{H}(a_i \mid i \in I)$. We say that a group in finitely generated if there exists a finite family $a_1, \ldots, a_r \in G$ such that $G = \mathrm{H}(a_1, \ldots, a_r)$. Finite groups are clearly finitely generated. The groups $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +_n)$ are generated by single elements, namely by 1 and $[1]_n$, respectively. Such groups are called cyclic groups. The groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^\times, \cdot)$ are not finite generated!

[15] Arthur Cayley (1821-1895) an English mathematician and leader of the British school of pure mathematics that emerged in the 19th century. The most important of Cayley's work is in developing the algebra of matrices and work in non-euclidean and n-dimensional geometry.

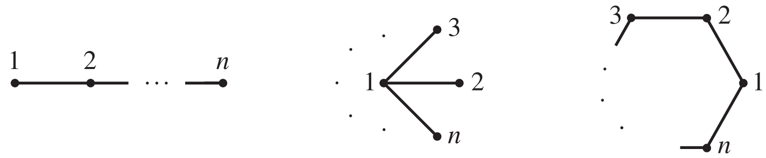
removing a edge at a time from a forest with $k-1$ root-trees $n-k+1$ forest with k root-trees) and hence $f_{n,k} = \binom{n-1}{k-1} n^{n-k}$, $1 \le k \le n$. – The required number is $f_{n,1}/n$.)

(d) The transpositions $\langle 1,2\rangle$, $\langle 2,3\rangle, \ldots, \langle n-1,n\rangle$ (respectively $\langle 1,2\rangle$, $\langle 1,3\rangle, \ldots, \langle 1,n\rangle$) form a minimal generating system of $\mathfrak{S}_n$. (**Hint :** If $a, b, c \in \{1,\ldots,n\}$ are three distinct elements, then we have $\langle a,b\rangle\langle a,c\rangle\langle a,b\rangle = \langle b,c\rangle$.)

(e) An analogous assertion to the part (a) also hold for the alternating group. For a "triangle" $\triangle = \{a,b,c\} \in \mathfrak{P}_3(\{1,\ldots,n\})$, let $\alpha(\triangle)$ denote the set of the two 3-cycles $\langle a,b,c\rangle$, $\langle a,c,b\rangle = \langle a,b,c\rangle^{-1}$ (which is independent of an order or of "orientation" of the $\triangle$). For 3-sets[16] $\triangle_1, \ldots, \triangle_m \in \mathfrak{P}_3(\{1,\ldots,n\})$, show that $\alpha(\triangle_1) \cup \cdots \cup \alpha(\triangle_m)$ generates the group $\mathfrak{A}(\Gamma_1) \times \cdots \times \mathfrak{A}(\Gamma_r) \subseteq \mathfrak{A}_n$, where $\Gamma_1, \ldots, \Gamma_r$ are the connected components of the graph with vertex-set $\{1,\ldots,n\}$ and whose edges belongs to any one of the triangle $\triangle_1, \ldots, \triangle_m$. (**Hint :** By induction on t prove that: If $\triangle_1, \ldots, \triangle_t$ are 3-sets with $\triangle_i \cap \triangle_{i+1} \neq \emptyset$ for $i = 1, \ldots, t-1$, then $\alpha(\triangle_1) \cup \cdots \cup \alpha(\triangle_t)$ generates the alternating group $\mathfrak{A}(\triangle_1 \cup \cdots \cup \triangle_t)$.) Deduce that: The minimal number of 3-cycles which generates the group $\mathfrak{A}_n$, $n \ge 3$, is $\lceil (n-1)/2 \rceil$. Give three 3-cycles which generates the group $\mathfrak{A}_5$, but no two (= $\lceil (5-1)/2 \rceil$) among them generate the group $\mathfrak{A}_5$.

(f) For $n \ge 3$, the set of 3-cycles $\langle 1,2,3\rangle$, $\langle 2,3,4\rangle, \ldots, \langle n-2,n-1,n\rangle$ (respectively, $\langle 1,2,3\rangle$, $\langle 1,2,4\rangle, \ldots, \langle 1,2,n\rangle$) form a generating system for the alternating group $\mathfrak{A}_n$.

†**T9.12** A permutation $\sigma \in \mathfrak{S}_n$ with s orbits has a representation as a product of $n-s$ transpositions and no representation as a product of less number of $n-s$ transpositions. (**Remark :** This Exercise has a natural generalisation: Let $T \subseteq \mathfrak{S}_n$ be a set of transpositions which generates the group $\mathfrak{S}_n$ (for example, by the given connected graph $\Gamma = \Gamma_T$ on the vertex set $\{1,\ldots,n\}$, see Test-Exercise 9.11-(a)). For $\sigma \in \mathfrak{S}_n$ determine the minimum $\ell(\sigma) = \ell_T(\sigma)$ of the $m \in \mathbb{N}$, for which there is a representation $\sigma = \tau_1 \cdots \tau_m$ with $\tau_i \in T$. Incidentally, $\ell(\sigma) = \ell(\sigma^{-1})$, and $d(\sigma_1,\sigma_2) := \ell(\sigma_2\sigma_1^{-1})$, $\sigma_1, \sigma_2 \in \mathfrak{S}_n$, is a metric on $\mathfrak{S}_n$. Further, the left- and right-translations ($\lambda_\tau : \mathfrak{S}_n \to \mathfrak{S}_n$, $\sigma \mapsto \tau\sigma$ and $\rho_\tau : \mathfrak{S}_n \to \mathfrak{S}_n$, $\sigma \mapsto \sigma\tau$) are distance preserving (enough to check that $d(\tau\sigma_1, \tau(\sigma_2) = \ell(\tau\sigma_2 \cdot (\tau\sigma_1)^{-1})\ell(\tau\sigma_2\sigma_1^{-1}\tau^{-1}) = \ell(\sigma_2\sigma_1^{-1}) = d(\sigma_1,\sigma_2)$ and similarly, $d(\sigma_1\tau, \sigma_2\tau) = d(\sigma_1,\sigma_2)$ for every *transposition* $\tau \in \mathfrak{S}_n$). For Γ_T, besides the complete graphs, one can also consider the following examples:



etc.

For the first of these graph see Exercise 9.2. For $T \subseteq T'$, it is clear that $\ell_{T'} \le \ell_T$.)

T9.13 **(a)** The cycles $\langle 1,2\rangle$, $\langle 2,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$, $n \ge 2$. (**Hint :** Use Test-Exercise 9.11-(d).)

(b) The cycles $\langle 1,2\rangle$, $\langle 1,2,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$, $n \ge 2$. More generally: if $k,n \in \mathbb{N}$ are natural numbers with $1 < k \le n$, then the cycles $\langle 1,k\rangle$, $\langle 1,2,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$ if and only if $\gcd(k-1,n) = 1$. (**Hint :** Use Test-Exercise 9.11-(d).)

(c) $\langle 1,n\rangle$, $\langle 1,\ldots,n\rangle$ generate the group $\mathfrak{S}_n$, $n \ge 2$. (**Hint :** Use Test-Exercise 9.11-(d).)

(d) If n is even (respectively, odd), then the cycles $\langle 1,2,3\rangle$, $\langle 1,2,3,\ldots,n\rangle$ (respectively $\langle 1,2,3\rangle$, $\langle 2,3,\ldots,n\rangle$) generate the alternating group $\mathfrak{A}_n$. (**Hint :** Use Test-Exercise 9.11-(f).)

T9.14 Let $n \in \mathbb{N}^+$. Show that

(a) The number of permutations $\tau \in \mathfrak{S}_n$ which commute with the permutation $\sigma \in \mathfrak{S}_n$ of the type $(\nu_1,\ldots,\nu_n)$ is $\nu_1! \cdots \nu_n!\, 1^{\nu_1} \cdots n^{\nu_n}$. (**Hint :** These permutations form the centraliser $\mathrm{C}_{\mathfrak{S}_n}(\sigma)$ of σ, see Example 9.A.20.)

[16]For any $r \in \mathbb{N}$, let $\mathfrak{P}_r(I)$ denote the subset of the power set $\mathfrak{P}(I)$ of a set I consisting of subsets $J \subseteq I$ of cardinality exactly r. With this r-s e t is an element $\mathfrak{P}_r(\{1,\ldots,n\})$, i. e. a subset of $\{1,\ldots,n\}$ of cardinality r.

(b) The number of involutions (= reflections) without any fixed point in $\mathfrak{S}_{2n}$ is $1\cdot 3\cdots(2n-1) = (2n)!/n!2^n$ ($\sim \sqrt{2}(2n/e)^n$ for $n\to\infty$).

(c) The number of involutions, i. .e. $\sigma^2 = \mathrm{id}$ (also called reflection) in $\mathfrak{S}_n$ is $\sum_{k\geq 0}\binom{n}{2k}\frac{(2k)!}{k!\,2^k}$.

(d) The number of permutations in $\mathfrak{S}_n$ with exactly t orbits is the *first Stirling's number* [17] $s(n,t)$.

(e) The number of permutations in $\mathfrak{S}_n$ such that its canonical decomposition contain a (and hence exactly one) cycle of length $> n/2$, is $n!\big(\sum_{n/2<k\leq n} 1/k\big)$ ($\sim n!\ln 2$ for $n\to\infty$).

(f) The number of permutations in $\mathfrak{S}_n$ without any fixed point is $n!\big(\sum_{k=0}^{n}(-1)^k/k!\big)$ ($\sim n!/e$ for $n\to\infty$).

T9.15 **(a)** Using the simplicity of the alternating group $\mathfrak{A}_n$, $n\geq 5$, prove that the group $\mathfrak{A}_n$ is the only non-trivial normal subgroup of the group $\mathfrak{S}_n$ for $n\geq 5$. (**Hint :** See Example 9.A.23.)

(b) Let $n\geq 2$ be a natural number. Show that the group $\mathfrak{S}_n$ is isomorphic to a subgroup of $\mathfrak{A}_{n+2}$, but not isomorphic to any subgroup of $\mathfrak{A}_{n+1}$.

T9.16 **(a)** The groups $\mathfrak{A}_4$ and $\mathfrak{V}_4$ are the only non-trivial normal subgroups in $\mathfrak{S}_4$.

(b) The group $\mathfrak{V}_4$ is the only non-trivial normal subgroup in $\mathfrak{A}_4$. (**Hint :** See Example 9.A.23.)

T9.17 **(a)** For a natural number $n\geq 2$, $\mathrm{Sign}:\mathfrak{S}_n\to\{-1,1\}$ is the only non-trivial group homomorphism. (**Hint :** $\langle ab\rangle$ and $\langle cd\rangle$ be two transpositions $\mathfrak{S}_n$. If $\sigma\in\mathfrak{S}_n$ be an arbitrary permutation with $a\mapsto c$, $b\mapsto d$, then $\sigma\langle ab\rangle\sigma^{-1} = \langle cd\rangle$ and so every homomorphism $\varphi:\mathfrak{S}_n\to\{1,-1\}$ have the same value on all transpositions. If this value is 1, then φ; if it is -1, then $\varphi = \mathrm{Sign}$.)

(b) Show that $\mathfrak{A}_n = [\mathfrak{S}_n,\mathfrak{S}_n]$(= the commutator subgroup[18] of $\mathfrak{S}_n$).

T9.18 Let I be a finite set and let $\sigma\in\mathfrak{S}(I)$ be a permutation of I whose order is a prime-power p^m, p a prime number. Show that the number of fixed points of σ and the cardinality $n := \#I$ of I are congruent modulo p. In particular, we have: (1) If n is not divisible by p, then σ has at least one fixed point. (2) If n is divisible by p, then the number of fixed points of σ is also divisible by p. (**Remark :** This is a special case of the assertion at the end of Example 6.E.5.)

T9.19 Which of the following maps $f:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ are bilinear, symmetric respectively alternating?

(a) $f\big((x_1,x_2),(y_1,y_2)\big) := x_1+y_2$. **(b)** $f\big((x_1,x_2),(y_1,y_2)\big) := x_1y_2$.

(c) $f\big((x_1,x_2),(y_1,y_2)\big) := x_1x_2-y_1y_2$. **(d)** $f\big((x_1,x_2),(y_1,y_2)\big) := x_1y_2-y_1x_2$.

(e) $f\big((x_1,x_2),(y_1,y_2)\big) := x_1y_2+y_1x_2$.

T9.20 Let V and W be K-vector spaces, I be a finite indexed set and $f:V^I\to W$ be a multi-linear map. Let $g:U\to V$ and $h:W\to X$ be K-vector space homomorphisms. Then $h\circ f\circ g^I:U^I\to X$ is a multi-linear map, where g^I is defined by $g^I\big((u_i)\big) := \big(g(u_i)\big)$, $(u_i)\in U^I$. If f is symmetric (respectively skew-symmetric, alternating), then so is $h\circ f\circ g^I$.

T9.21 Let v_j, $j\in J$ be a basis of the K-vector space V and let $w_{(j_i)}$, $(j_i)\in J^I$ be a family of elements of the K-vector space W, where I is a finite indexed set. Then there exists a unique

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[17] The Stirling's numbers $s(m,n)$, $0\leq n\leq m$, of first kind are defined by the equation: $\binom{x}{m} = \frac{1}{m!}\sum_{n=0}^{m}(-1)^{m-n}s(m,n)x^n$ (and otherwise $s(m,n)=0$). They clearly satisfy the recursion-formula: $s(0,n)=\delta_{0n}$ and $s(m+1,n) = ms(m,n)+s(m,n-1)$.

[18] For an arbitrary group G, the subgroup generated (see Footnote 14) by the commutators $[a,b] := aba^{-1}b^{-1}$, $a,b\in G$, is called the commutator subgroup or the derived group of G; it is usually denoted by $[G,G]$ or by $\mathrm{D}(G)$. Clearly, G is abelian if and only if $[G,G]$ is trivial. More generally, $[G,G]$ is a normal subgroup of G and the quotient group $G/[G,G]$ is abelian.

K-multi-linear map $f : V^I \to W$ such that $f\big((v_{j_i})_{i\in I}\big) = w_{(j_i)}$, $(j_i) \in J^I$. If V and W are finite dimensional, then the K-vector space of the multi-linear maps from V^I into W has the dimension $(\mathrm{Dim}_K V)^{|I|} \cdot \mathrm{Dim}_K W$.

T9.22 A n-linear map $f : V^n \to W$ of K-vector spaces is alternating if $f(x_1,\ldots,x_n) = 0$ for every n-tuple $(x_1,\ldots,x_n)$ in which two *consecutive* components are equal. (**Proof:** By induction on $d>0$, we shall show that $f(x_1,\ldots,x_n) = 0$ for all $i, j \in \{1,\ldots,n\}$ with $|i-j| = d$, if in the n-tuple $(x_1,\ldots,x_n)$ the i-th and the j-th components are equal. The case $d = 1$ is the hypothesis and so induction starts. For the inductive step we choose a $k \in \{1,\ldots,n\}$ in between i and j. Then $|i-k|$ and $|j-k|$ are smaller than d, and hence by the induction hypothesis

$$\begin{aligned} 0 = f(\ldots,x+y,\ldots,x+y,\ldots,x,\ldots) &= f(\ldots,x,\ldots,x,\ldots,x,\ldots) + f(\ldots,y,\ldots,x,\ldots,x,\ldots) \\ &+ f(\ldots,x,\ldots,y,\ldots,x,\ldots) + f(\ldots,y,\ldots,y,\ldots,x,\ldots) = f(\ldots,x,\ldots,y,\ldots,x,\ldots), \end{aligned}$$

where only the i-th, k-th and j-th components in the arguments are noted, the remaining are not altered.)

T9.23 Let K be a field and let V, W be vector spaces over K. Let $f : V^n \to K$ be an alternating multi-linear form on V and let $g : V \to W$ be a K-linear map. Show that the map

$$(x_0,\ldots,x_n) \longmapsto \sum_{i=0}^{n} (-1)^i f(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\, g(x_i)$$

is an alternating K-multi-linear map $V^{n+1} \to W$. (**Proof:** The map is obviously multi-linear. By Test-Exercise T9.22 it is enough to show that it vanish on every $(n+1)$-tuple with two equal consecutive components, say $x_i = x_{i+1} =: x$. Since f is alternating, in the above sum all terms except the i-th and the $(i+1)$-th term, are all 0. The remaining sum of two terms is:

$$\begin{aligned} &(-1)^i f(x_0,\ldots,x_{i-1},x_{i+1},x_{i+2},\ldots,x_n)\, g(x_i) + (-1)^{i+1} f(x_0,\ldots,x_{i-1},x_i,x_{i+2},\ldots,x_n)\, g(x_{i+1}) \\ &\quad = (-1)^i \big(f(x_0,\ldots,x_{i-1},x,x_{i+2},\ldots,x_n)\, g(x) - f(x_0,\ldots,x_{i-1},x,x_{i+2},\ldots,x_n)\, g(x)\big) = 0.) \end{aligned}$$

T9.24 Let A be a K-vector space of dimension n with a $(n+1)$-multi-linear map $A^{n+1} \to A$, $(x_0,\ldots,x_n) \mapsto x_0 \cdots x_{n+1}$. Then show that $\sum_{\sigma\in\mathfrak{S}_{n+1}} (\mathrm{Sign}\,\sigma)\, x_{\sigma 0} \cdots x_{\sigma n} = 0$ for all $x_0,\ldots,x_n \in A$. (**Hint :** By 9.B.7 the map $(x_0,\ldots,x_n) \mapsto \sum_{\sigma\in\mathfrak{S}_{n+1}} (\mathrm{Sign}\,\sigma)\, x_{\sigma 0} \cdots x_{\sigma n}$ is alternating $(n+1)$-linear map and by Corollary 9.B.6 it is 0, since $\mathrm{Dim} A = n$. – We mention the following example: Let $A \times A \to A$ be a K-bilinear (or an arbitrary) operation $(x,y) \mapsto xy$ on A. Then $\sum_{\sigma\in\mathfrak{S}_{n+1}} (\mathrm{Sign}\,\sigma)\, x_{\sigma 0} \cdots x_{\sigma n} = 0$ for all $x_0,\ldots,x_n \in A$, if we compute all the $(n+1)$-fold products with one and the same fixed given rule of parentheses. – There are $\frac{1}{n+1}\binom{2n}{n}$ possible rules of parentheses.)

T9.25 For the matrices

$$\mathfrak{A} := \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathfrak{B} := \begin{pmatrix} 5 & 5 & 3 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \end{pmatrix}$$

compute the adjoint matrices, the determinants and the product $\mathfrak{A} \cdot \mathrm{Adj}\,\mathfrak{A}$ and $\mathfrak{B} \cdot \mathrm{Adj}\,\mathfrak{B}$.

T9.26 Determine for which $a \in \mathbb{R}$ the following systems of linear equations over $\mathbb{R}$ has exactly one solution and in this case find the solution by the Cramer's rule:

$$\begin{aligned} ax_1 + x_2 + x_3 &= b_1 \\ x_1 + ax_2 + x_3 &= b_2 \\ x_1 + x_2 + ax_3 &= b_3\,, \end{aligned} \qquad\qquad \begin{aligned} x_1 + x_2 - x_2 &= b_1 \\ 2x_1 + 3x_2 + ax_2 &= b_2 \\ x_1 + ax_2 + 3x_2 &= b_3\,. \end{aligned}$$

T9.27 Let $\mathfrak{A} = (a_{ij})$ be an $n \times n$-matrix over the field K. For $c_1,\ldots,c_n \in K^\times$, show that $\mathrm{Det}\,(a_{ij}) = \mathrm{Det}\,(c_i c_j^{-1} a_{ij})$. In particular, $\mathrm{Det}\,(a_{ij}) = \mathrm{Det}\,\big((-1)^{i+j} a_{ij}\big)$.

T9.28 Let $\mathfrak{A}$ and $\mathfrak{B}$ be $n \times n$ invertible matrices over the field K. Then show that:

(a) $\mathrm{Adj}\,(\mathfrak{A}\mathfrak{B}) = \mathrm{Adj}\,\mathfrak{B} \cdot \mathrm{Adj}\,\mathfrak{A}$. **(b)** $\mathrm{Adj}\,\mathfrak{A}^{-1} = (\mathrm{Adj}\,\mathfrak{A})^{-1}$.

(c) $\operatorname{Det}(\operatorname{Adj}\mathfrak{A}) = (\operatorname{Det}\mathfrak{A})^{n-1}$. **(d)** $\operatorname{Adj}(\operatorname{Adj}\mathfrak{A}) = (\operatorname{Det}\mathfrak{A})^{n-2}\mathfrak{A}$.

(**Remark :** All these formulas, other than (b) are also valid for not-invertible matrices; for (d) assume $n > 1$).)

T9.29 Let $\mathfrak{A}$ be a non-invertible $n\times n$-matrix over the field K, $n \geq 1$. Show that the rank of the adjoint matrix $\operatorname{Adj}\mathfrak{A}$ is

$$\operatorname{Rank}\operatorname{Adj}\mathfrak{A} = \begin{cases} 1, & \text{if } \operatorname{Rank}\mathfrak{A} = n-1\,, \\ 0, & \text{if } \operatorname{Rank}\mathfrak{A} < n-1\,, \end{cases}$$

Moreover, if $\operatorname{Rank}\mathfrak{A} = n-1$, then show that every non-zero column of $\operatorname{Adj}\mathfrak{A}$ generates the kernel of $\mathfrak{A}$, i. e. the space of all $\mathfrak{x} \in K^n$ with $\mathfrak{A}\mathfrak{x} = 0$.

T9.30 The $n\times n$-matrix $\mathfrak{A}' = (a'_{ij})$ obtained from the $n\times n$-matrix $\mathfrak{A} = (a_{ij})$ by reflection through the anti-diagonal, i. e. $a'_{ij} = a_{n-j+1,n-i+1}$. Then show that $\operatorname{Det}\mathfrak{A}' = \operatorname{Det}\mathfrak{A}$, i. e.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{nn} & a_{n-1,n} & \cdots & a_{2n} & a_{1n} \\ a_{n,n-1} & a_{n-1,n-1} & \cdots & a_{1,n-1} & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n2} & a_{n-1,2} & \cdots & a_{22} & a_{12} \\ a_{n1} & a_{n-1,1} & \cdots & a_{21} & a_{11} \end{vmatrix}.$$

(**Hint :** Use $\operatorname{Det}\mathfrak{A} = \operatorname{Det}{}^{\mathrm{tr}}\mathfrak{A}$ and then the permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$, $i \mapsto n-i+1$ on the rows or columns of ${}^{\mathrm{tr}}\mathfrak{A}$ and use 9.D.2-(3) to conclude: $\operatorname{Det}{}^{\mathrm{tr}}\mathfrak{A}' = \operatorname{Det}\left(a'_{ij}\right) = \operatorname{Det}\left(a_{n-j+1,n-i+1}\right) = \operatorname{Det}\left(a_{n-j+1,\sigma(i)}\right) = \operatorname{Sign}(\sigma)\operatorname{Det}\left(a_{\sigma(j),i}\right) = \operatorname{Sign}(\sigma)\operatorname{Sign}(\sigma)\operatorname{Det}\left(a_{j,i}\right) = (\operatorname{Sign}(\sigma))^2\operatorname{Det}{}^{\mathrm{tr}}\mathfrak{A} = \operatorname{Det}\mathfrak{A}$.)

T9.31 Let $\mathfrak{A}$ and $\mathfrak{B}$ be $n\times n$-matrices with columns $x_1,\ldots,x_n \in K^n$ respectively, $y_1,\ldots,y_n \in K^n$. For a subset $J \subseteq \{1,\ldots,n\}$, let $\mathfrak{C}_J$ be the $n\times n$-matrix with the columns $z_1^{(J)},\ldots,z_n^{(J)}$, where

$$z_i^{(J)} := \begin{cases} x_i\,, & \text{if } i \in J\,, \\ y_i\,, & \text{if } i \notin J\,. \end{cases}$$

Show that

$$\operatorname{Det}(\mathfrak{A}+\mathfrak{B}) = \sum_{J\subseteq\{1,\ldots,n\}} \operatorname{Det}\mathfrak{C}_J\,.$$

(**Hint :** $\operatorname{Det}(\mathfrak{A}+\mathfrak{B}) = \Delta_{\mathfrak{e}}(x_1+y_1,\ldots,x_n+y_n) = \sum_J \Delta_{\mathfrak{e}}(z_1^{(J)},\ldots,z_n^{(J)}) = \sum_J \operatorname{Det}\mathfrak{C}_J$.)

T9.32 **(a)** Suppose that a column (or a row) of the $n\times n$-matrix $\mathfrak{A}$ has all entries 1. For the cofactors $(-1)^{i+j}A_{ij}$, $i,j = 1,\ldots,n$, of $\mathfrak{A}$, show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}A_{ij} = \operatorname{Det}\mathfrak{A}\,.$$

(b) Let $\mathfrak{A} = (a_{ij})$ be an $n\times n$-matrix over the field K with the cofactors $(-1)^{i+j}A_{ij}$, $i,j = 1,\ldots,n$. Further, let

$$\mathfrak{I} := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in \mathrm{M}_n(K)$$

is the matrix with all the coefficients are equal to 1. Show that

$$\operatorname{Det}(\mathfrak{A}+a\mathfrak{I}) = \operatorname{Det}\mathfrak{A} + a\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}A_{ij}\,.$$

T9.33 Let $\mathfrak{A}=(a_{ij})\in \mathrm{M}_n(\mathbb{Q})$ be an invertible matrix with integer coefficients a_{ij}. Show that the coefficients of the inverse matrix $\mathfrak{A}^{-1}$ are again integers if and only if $\mathrm{Det}\,\mathfrak{A}=\pm 1$.

T9.34 Let $\mathfrak{A}\in \mathrm{M}_n(K)$ be an upper-triangular matrix. Then show that $\mathrm{Adj}\,\mathfrak{A}$ and $\mathfrak{A}^{-1}$ (if $\mathfrak{A}$ is invertible) are also upper-triangular matrices.

T9.35 Let f_{ij}, $i,j=1,\ldots,n$ be differentiable functions on $D\subseteq \mathbb{K}$. Then show that

$$\begin{vmatrix} f_{11} & \cdots & f_{1n}\\ f_{21} & \cdots & f_{2n}\\ \vdots & \ddots & \vdots\\ f_{n1} & \cdots & f_{nn}\end{vmatrix}' = \begin{vmatrix} f'_{11} & \cdots & f'_{1n}\\ f_{21} & \cdots & f_{2n}\\ \vdots & \ddots & \vdots\\ f_{n1} & \cdots & f_{nn}\end{vmatrix} + \begin{vmatrix} f_{11} & \cdots & f_{1n}\\ f'_{21} & \cdots & f'_{2n}\\ \vdots & \ddots & \vdots\\ f_{n1} & \cdots & f_{nn}\end{vmatrix} + \cdots + \begin{vmatrix} f_{11} & \cdots & f_{1n}\\ f_{21} & \cdots & f_{2n}\\ \vdots & \ddots & \vdots\\ f'_{n1} & \cdots & f'_{nn}\end{vmatrix}.$$

T9.36 If $\sigma\in\mathfrak{S}(I)$ is a permutation of the finite indexed I and let

$$\mathfrak{P}_\sigma=(\delta_{i\sigma(j)})\in \mathrm{M}_I(K)$$

be the p e r m u t a t i o n m a t r i x a s s o c i a t e d t o σ. This is the matrix obtained from the unit matrix $\mathfrak{E}_I$ by permuting the columns according to σ: The j-th column of $\mathfrak{P}_\sigma$ is $e_{\sigma(j)}$, see Example 8.C.6. Then for $\sigma,\tau\in\mathfrak{S}(I)$: **(a)** $\mathrm{Det}\,\mathfrak{P}_\sigma=\mathrm{Sign}\,\sigma$. **(b)** $\mathfrak{P}_{\sigma\tau}=\mathfrak{P}_\sigma\mathfrak{P}_\tau$. **(c)** $(\mathfrak{P}_\sigma)^{-1}=\mathfrak{P}_{\sigma^{-1}}={}^t(\mathfrak{P}_\sigma)$.

T9.37 Let $\mathfrak{A}=(a_{ij})\in \mathrm{M}_I(K)$ be a s k e w - s y m m e t r i c matrix (I finite indexed), i.e. ${}^t\mathfrak{A}=-\mathfrak{A}$. If $|I|$ is odd and if $\mathrm{Char}\,K\neq 2$, i.e. $2=2\cdot 1_K\neq 0$ in K, then $\mathrm{Det}\,\mathfrak{A}=0$.

T9.38 Let $\mathfrak{A}:=\big(a_{ij}\big)\in \mathrm{M}_n(\mathbb{Z})$ be the $n\times n$-matrix defined by $a_{ij}:=\begin{pmatrix} i\\ j-1\end{pmatrix}$. Compute the determinant $\mathrm{Det}\,\mathfrak{A}$. (**Hint :** What is $a_{ij}-a_{i-1,j}$?)

T9.39 For two matrices $\mathfrak{A}\in \mathrm{M}_{m,n}(K)$ and $\mathfrak{B}\in \mathrm{M}_{n,m}(K)$ with $m>n$, show that $\mathrm{Det}\,(\mathfrak{A}\mathfrak{B})=0$. (**Hint :** Consider $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathrm{M}_{m,m}(K)$ by filling the extra entries 0.)

T9.40 Let K be a field and let $\mathfrak{A}\in \mathrm{M}_r(K)$, $\mathfrak{B}\in \mathrm{M}_s(K)$, $\mathfrak{C}\in \mathrm{M}_{r,s}(K)$. Then

$$\mathrm{Det}\begin{pmatrix}\mathfrak{C} & \mathfrak{A}\\ \mathfrak{B} & 0\end{pmatrix}=(-1)^{rs}\,\mathrm{Det}\,\mathfrak{A}\cdot\mathrm{Det}\,\mathfrak{B}.$$

T9.41 Prove the Multiplication-Theorem 9.D.5 for determinants as follows: Let $\mathfrak{A},\mathfrak{B}\in \mathrm{M}_n(K)$. By adding suitable multiples of the first n columns of the block-matrix

$$\begin{pmatrix}\mathfrak{A} & 0\\ -\mathfrak{E} & \mathfrak{B}\end{pmatrix}$$

to the last n columns transform this matrix to the matrix

$$\begin{pmatrix}\mathfrak{A} & \mathfrak{A}\mathfrak{B}\\ -\mathfrak{E} & 0\end{pmatrix}$$

and use Theorem 9.E.5 (and Test-Exercise T9.40).

T9.42 Let K be a field and let $\mathfrak{A}=\big(a_{ij}\big)\in \mathrm{M}_n(K)$, $n\in\mathbb{N}^*$ be a matrix of rank ≤ 1. Show that

$$\mathrm{Det}\,(a\mathfrak{E}+\mathfrak{A})=a^n+a^{n-1}\sum_{i=1}^n a_{ii}\quad\text{for all}\quad a\in K.$$

T9.43 Let $f_1,\ldots,f_n$ functions on the set D with values in the field K. Then show that $f_1,\ldots,f_n$ are linearly independent in K^D if and only if the function

$$(t_1,\ldots,t_n)\longmapsto\begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n)\\ \vdots & \ddots & \vdots\\ f_n(t_1) & \cdots & f_n(t_n)\end{vmatrix}$$

on D^n is not the zero-function. (**Remark :** See Theorem 5.G.17 – Determinants of this form are called alternant or (particularly in Physics) Slater's Determinant. For example the Vandermonde's determinant corresponding to $f_i := t^{i-1}$, $i = 1, \ldots, n$, $D := K$, see the Exercise 9.5-(a) and the Cauchy's double-alternants, see the Exercise 9.5-(b).)

T9.44 Let $f_1, \ldots, f_n$ be polynomial functions over K of deg $< n-1$, $n \in \mathbb{N}^*$. For all $t_1, \ldots, t_n \in K$, prove that

$$\begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} = 0\,.$$

T9.45 For $t_1, \ldots, t_n, u_1, \ldots, u_n \in \mathbb{C}$, compute

$$\begin{vmatrix} \sin(t_1+u_1) & \sin(t_1+u_2) & \cdots & \sin(t_1+u_n) \\ \sin(t_2+u_1) & \sin(t_2+u_2) & \cdots & \sin(t_2+u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(t_n+u_1) & \sin(t_n+u_2) & \cdots & \sin(t_n+u_n) \end{vmatrix}.$$

(**Hint :** The two cases $n \leq 2$ and $n > 2$ seperately. See also Test-Exercise T9.43.)

T9.46 Let D be a set, $t_1, \ldots, t_n \in D$ and $f_0, \ldots, f_n$ be linearly independent K-valued functions on D such that the $(n+1) \times n$-matrix

$$\begin{pmatrix} f_0(t_1) & \cdots & f_0(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{pmatrix}$$

has the maximal rank n. (because of the linear independence of $f_0, \ldots, f_n$, this is the case in general, see Test-Exercise T9.43. In this case we say that the points $t_1, \ldots, t_n$ are in in general position with respect to the $f_0, \ldots, f_n$.) Then show that the function

$$t \longmapsto \begin{vmatrix} f_0(t) & f_0(t_1) & \cdots & f_0(t_n) \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}$$

up to a uniquely determined constant factor $\lambda \neq 0$, is a non-trivial linear combination of the functions $f_0, \ldots, f_n$, which vanish on the points $t_1, \ldots, t_n$.

T9.47 Let D be a set, $E := \{t_1, \ldots, t_n\}$ be a subset of D with n elements and let $f_1, \ldots, f_n$ K-valued functions on D with

$$\begin{vmatrix} f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} \neq 0\,.$$

Show that the functions $f_1 \upharpoonright E, \ldots, f_n \upharpoonright E$ form a basis of K^E. For arbitrary elements $b_1, \ldots, b_n \in K$, there exists a unique linear combination f of $f_1, \ldots, f_n$ with $f(t_i) = b_i$, $i = 1, \ldots, n$. This follows from the equation

$$\begin{vmatrix} f(t) & b_1 & \ldots & b_n \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} = 0$$

by expanding in terms of the first column. (**Remark :** The uniquely determined function f is called the solution of the interpolation problem $f(t_i) = b_i$, $i = 1, \ldots, n$ with the functions $f_1, \ldots, f_n$.)

T9.48 **(a)** Let $P_i = (a_{1i}, \ldots, a_{ni})$, $i = 0, \ldots, n$ be points in the affine space $\mathbb{A}^n(K) = K^n$. Then the P_i are affinely dependent if and only if

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{vmatrix} .$$

(b) Let $P_i = (a_{1i}, \ldots, a_{ni})$, $i = 1, \ldots, n$ be affinely independent points in $\mathbb{A}^n(K) = K^n$. The equation of the affine hyperplane H in $\mathbb{A}^n(K)$ generated by the points $P_1, \ldots, P_n$ is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0 ,$$

i. e. the point $P = (x_1, \ldots, x_n) \in K^n$ belong to H if and only if its component satisfy the above (affine) equation. (See Test-Exercise T8.33.)

T9.49 Let $P_1 = (a_{11}, a_{21})$, $P_2 = (a_{12}, a_{22})$, $P_3 = (a_{13}, a_{23})$ be three points in $\mathbb{R}^2$ which do not lie on a line. Then show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & a_{11} & a_{12} & a_{13} \\ x_2 & a_{21} & a_{22} & a_{23} \\ x_1^2 + x_2^2 & a_{11}^2 + a_{21}^2 & a_{12}^2 + a_{22}^2 & a_{13}^2 + a_{23}^2 \end{vmatrix} = 0$$

is the equation of the circle passing through P_1, P_2, P_3.

T9.50 Let (a_{ij}) and (b_{ij}) be two $n \times n$-matrices over the field K. Then show that:

$$\sum_{i=1}^{n} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ b_{i1} & \cdots & b_{in} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}[\end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11} & \cdots & b_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{nj} & \cdots & a_{nn} \end{vmatrix} .$$

(**Hint:** If $(-1)^{i+j}A_{ij}$ are the cofactors of (a_{ij}), then by expanding the determinants by using the i-th row respectively the j-th column we have the equality:

$$\sum_{i=1}^{n} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ b_{i1} & \cdots & b_{in} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} (-1)^{i+j} b_{ij} A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} (-1)^{i+j} b_{ij} A_{ij} = \sum_{j=1}^{n} \begin{vmatrix} a_{11} & \cdots & b_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{nj} & \cdots & a_{nn} \end{vmatrix} .)$$

T9.51 Compute the following $n \times n$-determinants over $\mathbb{Q}$:

$$\begin{vmatrix} 1 & n & n & \cdots & n \\ n & 2 & n & \cdots & n \\ n & n & 3 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{vmatrix} , \qquad \begin{vmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 3 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & n \end{vmatrix} ,$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 2 & 3 & \cdots & n-1 \\ 3 & 2 & 1 & 2 & \cdots & n-2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n-1 & n-2 & n-3 & \cdots & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & 1 \\ 3 & 4 & 5 & \cdots & n & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & 1 & 2 & \cdots & n-3 & n-2 & n-1 \end{vmatrix}.$$

T9.52 Verify the following determinant formulas for $(n+1)\times(n+1)$-matrices with coefficients in a field K. (At the places marked by $*$ one may take arbitrary elements of K.)

(a)

$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix} = (a+nb)(a-b)^n.$$

(b) For $a_1,\ldots,a_n$, $b_1,\ldots,b_n$ be elements of K:

$$\begin{vmatrix} a_1 & * & * & * & \cdots & * & 1 \\ b_1 & a_2 & * & * & \cdots & * & 1 \\ b_1 & b_2 & a_3 & * & \cdots & * & 1 \\ b_1 & b_2 & b_3 & a_4 & \cdots & * & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & b_3 & b_4 & \cdots & a_n & * \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_n & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ b_1 & a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \\ * & b_2 & a_2 & a_2 & \cdots & a_2 & a_2 \\ * & * & b_3 & a_3 & \cdots & a_3 & a_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & a_{n-1} & a_{n-1} \\ * & * & * & * & \cdots & b_n & a_n \end{vmatrix} = (a_1-b_1)\cdots(a_n-b_n).$$

(c)

$$\begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 1 & a_1+b_1 & * & \cdots & * \\ 1 & a_1 & a_2+b_2 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix} = b_1\cdots b_n.$$

(d)

$$\begin{vmatrix} -a_1 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & a_2 & \cdots & 0 & 0 \\ 0 & 0 & -a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_n & a_n \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = (-1)^n(n+1)\,a_1\cdots a_n.$$

T9.53 Prove the following determinant formulas for the $n\times n$-matrices over a field K: Let $a_1,\ldots,a_n$, $b_1,\ldots,b_n$, $c_1,\ldots,c_{n-1}$ be elements of K and let

$$D_n := \begin{vmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{vmatrix}$$

(a) (Recursion formula): $D_k = a_k D_{k-1} - b_{k-1}c_{k-1}D_{k-2}$, for all $k = 2, \ldots, n$.

(b) In part (a) put $b_1 = \cdots = b_{n-1} = c_1 = \cdots c_{n-1} =: b$ and $D_n := D(b; a_1, \ldots, a_n)$. Then

$$D(b;a_1,\ldots,a_n) = a_n D(b;a_1,\ldots,a_{n-1}) - b^2 D(b;a_1,\ldots,a_{n-2}) \text{ for all } n \geq 2.$$

(c) Compute the determinant $D(b;a_1,\ldots,a_n)$ in the following cases: (1) $b = a_1 = \cdots = a_n = 1$. (2) $a_1 = \cdots = a_n = 0$. (3) $K = \mathbb{K}$ and $b = 1$, $a_1 = \cos\varphi$, $a_2 = \cdots = a_n = 2\cos\varphi$.

$$\begin{vmatrix} \cos\varphi & 1 & 0 & \cdots & 0 \\ 1 & 2\cos\varphi & 1 & \cdots & 0 \\ 0 & 1 & 2\cos\varphi & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\varphi \end{vmatrix} = \cos n\varphi\,, \quad \varphi \in \mathbb{C}.$$

(**Remark:** For the modified Tchebychev Polynomial $\widetilde{T}_n$ see the recursion-formula in (3)-(iii) below. – Recall the definition and some properties of **Tchebychev Polynomials:** For $n \in \mathbb{N}$ the polynomials

$$T_n(X) := \sum_{k=0}^{[n/2]} \left(-\frac{1}{4}\right)^k \frac{n}{n-k} \binom{n-k}{k} X^{n-2k} \text{ and } U_n(X) := \sum_{k=0}^{[n/2]} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} X^{n-2k}$$

are called Tchebychev polynomials of first and second kind respectively.

Properties of Tchebychev polynomials.

1) $T_0 = 2, T_1 = X$ and $T_{n+2} = XT_{n+1} - \frac{1}{4}T_n$ for every $n \in \mathbb{N}$.

2) $2^{n-1}T_n(\cos(\varphi)) = \cos(n\varphi)$ for every $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$.

3) For $n \in \mathbb{N}$, put $\widetilde{T}_n(X) := 2^{n-1}T_n(X)$. Then:

(i) $\widetilde{T}_0 = 1, \widetilde{T}_1 = X$ and $\widetilde{T}_{n+2} = 2X\widetilde{T}_{n+1} - \widetilde{T}_n$ for every $n \in \mathbb{N}$.

(ii) Let $n \in \mathbb{N}$. Then $\widetilde{T}_n(1) = 1, \widetilde{T}_n(-1) = (-1)^n$ and $\widetilde{T}_n(0) = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

(iii) $\widetilde{T}_n(\cos(\varphi)) = \cos(n\varphi)$ for every $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$.

4) T_n and $\widetilde{T}_n$ have n-distinct real zeros in the open interval $(-1,1)$, namely : $\cos((2k+1)\pi/2n)$ for $k = 0, \ldots, n-1$ and therefore $T_n(X) = \prod_{k=0}^{n-1}\Big(X - \cos((2k+1)\pi/2n)\Big)$ for every $n \geq 1$.

5) $U_0 = 1, U_1 = X$ and $U_{n+2} = XU_{n+1} - \frac{1}{4}U_n$ for every $n \in \mathbb{N}$.

6) $2^{n-1}U_{n-1}(\cos(\varphi)) = \dfrac{\sin(n\varphi)}{\sin(\varphi)}$ for every $n \in \mathbb{N}^+$ and $\varphi \in \mathbb{R}$, with $\varphi \notin \mathbb{Z}\pi$.

7) Let $n \in \mathbb{N}$. Then $U_n(X) = \prod_{k=1}^{n}\Big(X - \cos((k\pi)/(n+1))\Big)$ and $U_{2n}(X) = \prod_{k=1}^{n}\Big(X^2 - \cos^2((k\pi)/(2n+1))\Big)$. In particular, $n+1 = 2^nU_n(1) = 2^n \cdot \prod_{k=1}^{n}\Big(1 - \cos((k\pi)/(n+1))\Big)$ and $2n+1 = 2^{2n}U_{2n}(1) = 2^{2n} \cdot \prod_{k=1}^{n}\Big(1 - \sin^2((k\pi)/(2n+1))\Big)$ for every $n \geq 1$.)

(4) $a_1 = \cdots = a_n =: a$.

$$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & b & a \end{vmatrix} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} a^{n-2k} b^{2k}.$$

For $a=2$, $b=1$ and for $a=b=1$ compute the value of this determinant directly and verify this with the given sum-formula.

(d) In part (a) put $b_1=\cdots=b_{n-1}=-c_1=\cdots-c_{n-1}=:b$ and $D_n:=\Delta(b;a_1,\ldots,a_n)$. Then
$$\Delta(b;a_1,\ldots,a_n)=a_n\Delta(b;a_1,\ldots,a_{n-1})-b^2\Delta(b;a_1,\ldots,a_{n-2}) \text{ for all } n\geq 2.$$
Further, for $a_1=\cdots=a_n=:a$,
$$\Delta(b;a,\ldots,a)=\begin{vmatrix} a & b & 0 & \cdots & 0 & 0\\ -b & a & b & \cdots & 0 & 0\\ 0 & -b & a & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a & b\\ 0 & 0 & 0 & \cdots & -b & a\end{vmatrix}=\sum_{k=0}^{[n/2]}\binom{n-k}{k}a^{n-2k}b^{2k}.$$
For $a=b=1$, this determinant $\Delta(1;1,\ldots,1)$ is the Fibonacci-number[19] f_{n+1}.

(e) Compute the determinants of the following matrices in $\mathrm{M}_n(\mathbb{Z})$:
$$\begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0\\ 1 & 2 & 1 & \cdots & 0 & 0 & 0\\ 0 & 1 & 2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 2 & 1\\ 0 & 0 & 0 & \cdots & 0 & 1 & 2\end{vmatrix},\quad \begin{vmatrix} 1 & 1^2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ -1 & 1 & 2^2 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 3^2 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & (n-2)^2 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 & (n-1)^2\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1\end{vmatrix}$$
(**Hint :** Use induction on n. See also part (c).)

(f)
$$\begin{vmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0\\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2\end{vmatrix}$$

T9.54 Let $a_1,\ldots,a_n,b$ and a_{ij}, $1\leq i,j\leq n$ be elements of a field K. Then show that

(a)
$$\begin{vmatrix} a_0+a_1 & a_1 & 0 & \cdots & 0\\ a_1 & a_1+a_2 & a_2 & \cdots & 0\\ 0 & a_2 & a_2+a_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & a_{n-1}+a_n\end{vmatrix}=\sum_{k=0}^{n}\Big(\prod_{i\neq k}a_i\Big).$$

[19]**Fibonacci-numbers.** The sequence $(f_n)_{n\in\mathbb{N}}$ of integers which is defined recursively as : $f_0=0, f_1=1, f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$ is called the Fibonacci sequence and its n-th term f_n is called the n-th Fibonacci number. First few terms of the Fibonacci sequence are $0,1,1,2,3,5,8,13,21,34,55,\ldots$ For the n-th Fibonacci number there is an explicit formula (Binet's formula): $f_n:=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

(b)

$$\begin{vmatrix} a_{11}+b & a_{12}+b & \cdots & a_{1n}+b \\ a_{21}+b & a_{22}+b & \cdots & a_{2n}+b \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b & a_{n2}+b & \cdots & a_{nn}+b \end{vmatrix} = a+b\left(\sum_{i,j=1}^{n} a'_{ij}\right),$$

where $a := \operatorname{Det}\big(a_{ij}\big)$ and a'_{ij} is the (i,j)-th cofactor of $\big(a_{ij}\big)$, $1 \le i,j \le n$.

T9.55 Prove the following determinant formulas by induction:

(a)

$$\begin{vmatrix} a_1+b_1 & b_1 & b_1 & \cdots & b_1 \\ b_2 & a_2+b_2 & b_2 & \cdots & b_2 \\ b_3 & b_3 & a_3+b_3 & \cdots & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & b_n & b_n & \cdots & a_n+b_n \end{vmatrix} = a_1\cdots a_n + \sum_{k=1}^{n}\Big(\prod_{i\neq k} a_i\Big) b_k \,,$$

(b)

$$\begin{vmatrix} x+a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 0 \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix} = x^n + a_1x^{n-1} + \cdots + a_n \,,.$$

(c)

$$\begin{vmatrix} a_1 & \cdots & 0 & 0 & \cdots & b_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_n & b_n & \cdots & 0 \\ 0 & \cdots & b_n & a_n & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_1 & \cdots & 0 & 0 & \cdots & a_1 \end{vmatrix} = \prod_{k=1}^{n}(a_k^2 - b_k^2)\,.$$

T9.56 Suppose that the matrix $\mathfrak{A} = (a_{ij}) \in \mathrm{GL}_n(K)$ satisfy the hypothesis of Test-Exercise T8.42 and suppose that $\mathfrak{A} = \mathfrak{L}\mathfrak{D}\mathfrak{R}'$ with a diagonal matrix $\mathfrak{D} = \operatorname{Diag}(a_1,\ldots,a_n)$ and a normalised lower respectively upper triangular matrix $\mathfrak{L}$ respectively $\mathfrak{R}'$. Then $a_k = D_k/D_{k-1}$, $k = 1,\ldots,n$, where $D_k = \operatorname{Det}(a_{ij})_{1\le i,j\le k}$ is the k-th principal minor of $\mathfrak{A}$, $k = 0,\ldots,n$. (Put $D_0 = 1$.)

T9.57 Let $n \in \mathbb{N}^*$ and let K be a field. The canonical exact sequence

$$1 \longrightarrow \mathrm{SL}_n(K) \longrightarrow \mathrm{GL}_n(K) \xrightarrow{\ \mathrm{Det}\ } K^\times \longrightarrow 1$$

is a weak-split. Further, it is strong-split if and only if the power-map $x \mapsto x^n$ is an automorphism of $K^\times$. (**Remarks:** An *exact sequence* (i. e. (i) φ is injective, (ii) ψ is surjective and (iii) $\operatorname{Im}\varphi = \operatorname{Ker}\psi$.)

(*) $$1 \to N \xrightarrow{\ \varphi\ } G \xrightarrow{\ \psi\ } H \to 1$$

of groups (not necessary abelian) is called a w e a k s p l i t s e q u e n c e if ψ has a *section* σ, i. e. there exists a homomorphism $\sigma : H \to G$ such that $\psi\sigma = \mathrm{id}_H$ (this means G is the *semi-direct product* of $\operatorname{Im}\varphi \cong N$ and $\operatorname{Im}\sigma \cong H$) and $\operatorname{Im}\sigma$ is called a w e a k c o m p l e m e n t of $\operatorname{Im}\varphi$ in G. – If there exists a projection $\pi : G \to N$ such that $\pi\varphi = \mathrm{id}_N$, then G is a direct product of $\operatorname{Im}\varphi \cong N$ and $\operatorname{Ker}\pi \cong H$, i. e. the map $\operatorname{Im}\varphi \times \operatorname{Ker}\pi \to G$, $(x,y) \mapsto xy$ is an isomorphism of groups. In this we say that the exact sequence $(*)$ is a s t r o n g s p l i t s e q u e n c e and $\operatorname{Ker}\pi$ is called a s t r o n g c o m p l e m e n t of $\operatorname{Im}\varphi$ in G. – Every strong split sequence is a weak split sequence. If σ is a section of ψ and if $\operatorname{Im}\sigma$ is a *normal* in G, then $\operatorname{Im}\sigma$ is a strong complement if $\operatorname{Im}\varphi$ in G and the exact sequence $(*)$ is a strong split. – If G (and hence H and N are abelian) then an exact sequence $(*)$ is weak split if and only if its strong split.)

T9.58 Let $f\colon V\to V$ be a nilpotent endomorphism of the n-dimensional K-vector space V. Then show that $\operatorname{Det}(a\,\mathrm{id}_V+f)=a^n$ for all $a\in K$. More generally, show that $\operatorname{Det}(g+f)=\operatorname{Det}g$ for every operator g on V which commute with f, i. e. $gf=fg$.

T9.59 Let $V:=K[t]$ be the vector space of all polynomial functions over the infinite field K and let $V_n:=K[t]_n$ be the subspace of all polynomial functions of degree $<n$, $n\in\mathbb{N}^*$.

(a) For $a,b\in K$, let $\varepsilon\colon V\to V$ be defined by $f(t)\mapsto f(at+b)$. Show that ε linear and $\varepsilon(V_n)\subseteq V_n$ for all n. Further, compute the determinant $\operatorname{Det}(\varepsilon|V_n)$.

(b) Let $K=\mathbb{K}$. For $c_0,\ldots,c_r\in\mathbb{K}$, let $\delta\colon V\to V$ be the differential operator

$$f(t)\mapsto\sum_{k=0}^{r}c_kf^{(k)}(t)\,.$$

Show that δ linear and for every $n\in\mathbb{N}^*$, $\delta(V_n)\subseteq V_n$. Further, compute the determinant $\operatorname{Det}(\delta|V_n)$.

T9.60 Let $m,n\in\mathbb{N}$ with $m\le n$. For arbitrary matrices $\mathfrak{A}=\big(a_{ij}\big)\in\mathrm{M}_{m,n}(K)$ and $\mathfrak{B}=\big(b_{ji}\big)\in\mathrm{M}_{n,m}(K)$ over a field K, show that

$$\operatorname{Det}(\mathfrak{A}\mathfrak{B})=\sum_{1\le j_1<\cdots<j_m\le n}\begin{vmatrix}a_{1,j_1}&\cdots&a_{1,j_m}\\\vdots&\ddots&\vdots\\a_{m,j_1}&\cdots&a_{m,j_m}\end{vmatrix}\cdot\begin{vmatrix}b_{j_1,1}&\cdots&b_{j_1,m}\\\vdots&\ddots&\vdots\\b_{j_m,1}&\cdots&b_{j_m,m}\end{vmatrix}$$

(**Hint :** Let $f:K^n\to K^m$ and $g:K^m\to K^n$ be the linear maps defined by the matrices $\mathfrak{A}$ and $\mathfrak{B}$ (with respect to the standard bases), respectively. Then compute the composition $\operatorname{Alt}(m,f\circ g)=\operatorname{Alt}(m,g)\circ\operatorname{Alt}(m,f)$ using the basis Δ_H, $H\in\mathfrak{P}_m(\{1,\ldots,n\})$ of the K-vector space $\operatorname{Alt}(m,K^n)$.)

T9.61 (N o r m) Let A be a finite dimensional K-algebra. For $x\in A$, let $\lambda_x\colon A\to A$ be the left-multiplication $y\mapsto xy$ by x on A. Show that λ_x is a K-linear operator on A. Its determinant is called the N o r m o f x (over K) and is denoted by $\mathrm{N}_K^A(x)=\mathrm{N}(x)$.

(a) For all $x,y\in A$, $\mathrm{N}(xy)=\mathrm{N}(x)\mathrm{N}(y)$.

(b) For all $a\in K$, $\mathrm{N}(a):=\mathrm{N}(a\cdot 1_A)=a^n$, $n:=\operatorname{Dim}_KA$.

(c) An element $z\in A$ is a unit in A if and only if $\mathrm{N}(x)\neq 0$ in K.

T9.62 For all elements z of the $\mathbb{R}$-Algebra $\mathbb{C}$, show that $\mathrm{N}_{\mathbb{R}}^{\mathbb{C}}(z)=|z|^2$. (**Hint :** see Test-Exercise T9.61.)

T9.63 Let $A=\mathrm{M}_n(K)$ be the algebra of $n\times n$-matrices over the field K. For all $\mathfrak{A}\in A$, show that $\mathrm{N}_K^A(\mathfrak{A})=(\operatorname{Det}\mathfrak{A})^n$ (**Hint :** see, Test-Exercise T9.61. – One can use the least computation by using: $\mathrm{N}_K^A(\mathfrak{A})=(\operatorname{Det}\mathfrak{A})^m$ for a fixed $m\in\mathbb{N}$. Compute this m by specialising the matrix $\mathfrak{A}$, see 9.D.9.)

T9.64 Let V be a finite dimensional $\mathbb{C}$-vector space and let $f\colon V\to V$ be a $\mathbb{C}$-linear operator on V. We consider V as a $\mathbb{R}$-vector space, then f is a $\mathbb{R}$-linear operator and its determinant is denoted by $\operatorname{Det}_{\mathbb{R}}f$. Show that $\operatorname{Det}_{\mathbb{R}}f=|\operatorname{Det}f|^2$. (**Hint:** If $\mathfrak{A}+\mathrm{i}\,\mathfrak{B}$, $\mathfrak{A},\mathfrak{B}\in\mathrm{M}_n(\mathbb{R})$, is the matrix of f with respect to the $\mathbb{C}$-Basis $v_1,\ldots,v_n$ of V, then

$$\begin{pmatrix}\mathfrak{A}&-\mathfrak{B}\\\mathfrak{B}&\mathfrak{A}\end{pmatrix}\in\mathrm{M}_{2n}(\mathbb{R})$$

is the matrix of f with respect to the $\mathbb{R}$-Basis $v_1,\ldots,v_n$, $\mathrm{i}v_1,\ldots,\mathrm{i}v_n$ and

$$\begin{vmatrix}\mathfrak{A}&-\mathfrak{B}\\\mathfrak{B}&\mathfrak{A}\end{vmatrix}=\begin{vmatrix}\mathfrak{A}-\mathrm{i}\,\mathfrak{B}&-\mathfrak{B}\\\mathfrak{B}+\mathrm{i}\mathfrak{A}&\mathfrak{A}\end{vmatrix}=\begin{vmatrix}\mathfrak{A}-\mathrm{i}\,\mathfrak{B}&-\mathfrak{B}\\0&\mathfrak{A}+\mathrm{i}\,\mathfrak{B}\end{vmatrix}\,.)$$

In particular, if A is a finite dimensional $\mathbb{C}$-algebra, then for all $x\in A$ (see Test-Exercise T9.61) show that

$$\mathrm{N}_{\mathbb{R}}^A(x)=|\mathrm{N}_{\mathbb{C}}^A(x)|^2\,.$$

T9.65 Determine which of the following affinities of an n-dimensional oriented real affine spaces are orientation preserving: **(a)** point-reflections. **(b)** reflections of a hyperplanes along a lines and product of such r reflections, $r \in \mathbb{N}$. **(c)** transvections. **(d)** dilatations. **(e)** magnifications.

T9.66 Let E be an oriented n-dimensional $\mathbb{R}$-affine space. Suppose that the affine basis $P_0, \ldots, P_n$ represents the orientation of E. For a permutation σ in $\mathfrak{S}(\{0, \ldots, n\})$, show that the affine basis $P_{\sigma(0)}, \ldots, P_{\sigma(n)}$ represents the orientation of E if and only if σ is even. Further, show that the affine Basis $P_n, \ldots, P_0$ also represents the orientation of E if and only if $n \equiv 0$ or $n \equiv 3$ modulo 4. (**Hint :** See also Exercise 9.9-(a).)

T9.67 In every subgroup of the affine group $\mathrm{A}(E)$ of an oriented finite dimensional real affine space E which has at least one orientation reversing map, the subset of all orientation preserving maps form a subgroup of index 2.

T9.68 Suppose that the finite dimensional $\mathbb{R}$-vector space V is the direct sum of the subspaces U and W. By the following specifications of orientations on two of the spaces U, V, W a orientation on the third is determined: Suppose that $\mathfrak{u} = (u_1, \ldots, u_r)$ respectively $\mathfrak{w} = (w_1, \ldots, w_s)$ are bases of U respectively W. Then the basis $(u_1, \ldots, u_r, w_1, \ldots, w_s)$ represents the orientation of $V = U \oplus W$ if and only if the bases $\mathfrak{u}$ respectively $\mathfrak{w}$ both represents (or both don't represent) the orientations of U and W respectively. (**Hint :** Note the dependence on the sequence U and W.)

T9.69 Let V be a finite dimensional $\mathbb{R}$-vector space and let $V' \subseteq V$ be a subspace with the corresponding quotient space $\overline{V} = V/V'$. By the specifications of the orientations on the two of the spaces $V', V, \overline{V}$ a orientation on the third is determined: Suppose that $v'_1, \ldots, v'_r \in V'$ is a basis of V' and that the residue-classes of $v_1, \ldots, v_s \in V$ form a basis of $\overline{V}$. Show that the basis $v'_1, \ldots, v'_r, v_1, \ldots, v_s$ of V represents the orientation of V if and only if the bases $v'_1, \ldots, v'_r$ of V' and $\overline{v}_1, \ldots, \overline{v}_s$ of $\overline{V}$ both represent (or both don't represent) the orientations of V' and $\overline{V}$ respectively.

T9.70 Determine which of the following bases of $\mathbb{R}^n$ represent the standard orientation:

(a) $n = 2$; $v_1 = (1, 1)$, $v_2 = (1, -1)$.

(b) $n = 3$; $v_1 = (-1, 0, 1)$, $v_2 = (0, -1, 1)$, $v_3 = (1, -1, 1)$.

(c) $n = 4$; $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 2, 1, 1)$, $v_3 = (1, 1, 3, 1)$, $v_4 = (1, 1, 1, 4)$.

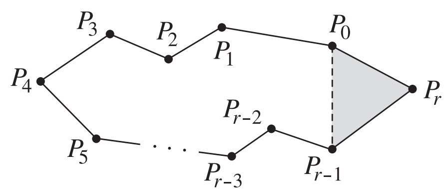
T9.71 **(a)** Every $\mathbb{C}$-linear isomorphism of finite dimensional complex vector spaces is orientation preserving. (see Example 9.F.6)

(b) A $\mathbb{C}$-anti-linear isomorphism of finite dimensional complex vector spaces (see Example 5.C.7) is orientation preserving if and only if their common complex dimension is even.

T9.72 Let E be a real affine plane with the volume-function $\lambda_{\mathfrak{v}}$ with respect to the basis v_1, v_2 of the space of the translations of E and $P_0, \ldots, P_r$, $r \geq 2$, be points with the coordinates (a_j, b_j), $j = 0, \ldots, r$, with respect to an affine coordinate system $O; v_1, v_2$. Furthermore, let $[P_0, P_1, \ldots, P_r, P_0]$ be a simple closed polygon, i. e. the edges meet exactly at the adjacent vertices. Show that the surface area of enclosed polygon is, up to a sign, equal to

$$\frac{1}{2}\left(\mathrm{Det}\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} + \cdots + \mathrm{Det}\begin{pmatrix} a_{r-1} & a_r \\ b_{r-1} & b_r \end{pmatrix} + \mathrm{Det}\begin{pmatrix} a_r & a_0 \\ b_r & b_0 \end{pmatrix}\right).$$

(**Remark:** What do we mean by sign? Think about the orientation of E. – For the inductive-step from $r-1$ to r use: by suitable numbering of the vertices of the polygon with vertices $P_0, \ldots, P_{r-1}$ and the complement of the triangle with the vertices P_{r-1}, P_r, P_0 with only one common edge $[P_{r-1}, P_0]$.)



T9.73 Let $f_1,\ldots,f_n$ be a basis of the space of linear forms on $\mathbb{R}^n$. Let $\mathfrak{A} := (a_{ij}) \in \mathrm{GL}_n(\mathbb{R})$ be the transition matrix from the dual basis $e_1^*,\ldots,e_n^*$ (with respect to the standard basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$) to the basis $f_1,\ldots,f_n$. Therefore $f_j = \sum_{i=1}^n a_{ij}e_i^*$, and $f_1,\ldots,f_n$ is the dual basis with respect to the basis $v_j = \sum_{i=1}^n b_{ij}e_i$, $j=1,\ldots,n$, where $\mathfrak{B} := (b_{ij}) = {}^t\mathfrak{A}^{-1}$ is the contra-gradient matrix of $\mathfrak{A}$ (see Test-Exercise T8.20). Let $d := |\mathrm{Det}\,\mathfrak{A}|$. Show that

(a) For $c_1,\ldots,c_n \geq 0$, the volume of $\{x \in \mathbb{R}^n \mid |f_i(x)| \leq c_i\,,\ i=1,\ldots,n\}$ is equal to $2^n c_1\cdots c_n/d$.

(b) For $c \geq 0$, the volume of $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n |f_i(x)| \leq c\}$ is equal to $2^n c^n/n!\,d$.

(c) For $c \geq 0$, the volume of the ellipsoid $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n |f_i(x)|^2 \leq c^2\}$ is equal to $\omega_n c^n/d$, where ω_n have the same meaning as in Exercise 9.9-(a).

(d) For $c_0,c_1,\ldots,c_n \in \mathbb{R}$ with $c_0 \leq c_1+\cdots+c_n$, the volume of the simplex

$$\{x \in \mathbb{R}^n \mid f_i(x) \leq c_i\,,\ i=1,\ldots,n\,,\ f_1(x)+\cdots+f_n(x) \geq c_0\}$$

is equal to $b^n/n!\,d$ mit $b := c_1+\cdots+c_n-c_0$. (**Proof:** The matrix of the linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with $f(x_1,\ldots,x_n) = \big(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\big)$ with respect to the staandard basis is the transpose ${}^{\mathrm{tr}}\mathfrak{A}$. Therefore $\mathrm{Det} f = \mathrm{Det}\,{}^{\mathrm{tr}}\mathfrak{A} = \mathrm{Det}\,\mathfrak{A} = d$ and so $|\mathrm{Det} f^{-1}| = d^{-1}$. Now by Theorem 9.G.2 and the remarks after that $\lambda^n\big(f^{-1}(M)\big) = \lambda^n(M)/d$. for every set M for which $\lambda^n(M)$ is defined.

(a) The volume of the cuboid $Q := [-c_1,c_2]\times\cdots\times[-c_n,c_n]$ is equal to the product $(2c_1)\cdots(2c_n) = 2^n c_1\cdots c_n$ of the lengths of its sides, and it follows that $\lambda^n(Q) = \lambda^n\big(\{x\in\mathbb{R}^n \mid |f_1(x)| \leq c_1,\ldots,|f_n(x)| \leq c_n\}\big) = \lambda^n\big(f^{-1}\big([-c_1,c_2]\times\cdots\times[-c_n,c_n]\big)\big) = 2^n c_1\cdots c_n/d$.

(b) Since the volume of the simplex $\{y=(y_1,\ldots,y_n)\in\mathbb{R}_+^n \mid y_1+\cdots+y_n \leq c\}$ (by 9.G.4) is equal to $c^n/n!$, the volume of $M := \{y=(y_1,\ldots,y_n)\in\mathbb{R}^n \mid |y_1|+\cdots+|y_n| \leq c\}$ is $2^n c^n/n!$. It follows that $\lambda^n(M) = \lambda^n\big(\{x\in \mathbb{R}^n \mid |f_1(x)|+\cdots+|f_n(x)| \leq c\}\big) = \lambda^n\big(f^{-1}(M)\big) = 2^n c^n/dn!$.)

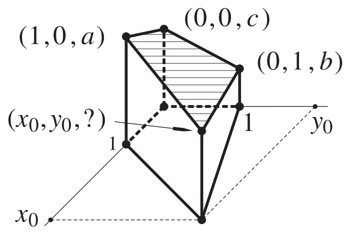
T9.74 Let $P_0,\ldots,P_n \in \mathbb{R}^n$ be affinely independent points and let S be the (convex) simplex with these vertices. Further, let $y_0,\ldots,y_n \in \mathbb{R}_+$ and H be the affine hyperplane in $\mathbb{R}^{n+1}$ through the points $(P_0,y_0),\ldots,(P_n,y_n) \in \mathbb{R}^{n+1}$. Therefore H is the graph of the affine function $h\colon\mathbb{R}^n \to \mathbb{R}$ with $h(P_i) = y_i$, $i=0,\ldots,n$. If $T \subseteq \mathbb{R}^{n+1}$ is the solid-body in between S and H, i. e.

$$T := \big\{(x,y) \in \mathbb{R}^{n+1} \bigm| x \in S\,,\ 0 \leq y \leq h(x)\big\}\,,$$

then

$$\lambda^{n+1}(T) = \frac{y_0+\cdots+y_n}{n+1}\,\lambda^n(S)\,.$$

(**Hint:** $\lambda^{n+1}(T)$ is additive in $(y_0,\ldots,y_n)$ and does not change if the values $y_0,\ldots,y_n$ are permutated. One can also assume that all y_i are equal or that all y_i other than a value y_{i_0} vanish.) Compute the volume of the following solid-bodies in $\mathbb{R}^3$, where the top surface area is:



T9.75 The group $\mathrm{GL}_n(\mathbb{R})$, $n \in \mathbb{N}^*$, is the direct product of the groups $\mathrm{I}_n(\mathbb{R})$ of volume preserving (or u n i m o d u l a r) matrices $\mathfrak{B} \in \mathrm{GL}_n(\mathbb{R})$ with $|\mathrm{Det}\mathfrak{B}| = 1$ and the group $\mathbb{R}_+^\times \mathfrak{E}_n \cong \mathbb{R}_+^\times$ of the scalar matrices $a\mathfrak{E}_n$, $a \in \mathbb{R}_+^\times$, i. e. every matrix $\mathfrak{A} \in \mathrm{GL}_n(\mathbb{R})$ has a representation $\mathfrak{A} = a\mathfrak{B} = \mathfrak{B}a$ with uniquely determined (by $\mathfrak{A}$) elements $a \in \mathbb{R}_+^\times$ and $\mathfrak{B} \in \mathrm{I}_n(\mathbb{R})$. (**Remark :** Therefore, every linear automorphism f of $\mathbb{R}^n$ is the composition of a volume-preserving automorphism g and a homothecy $a\cdot\mathrm{id}$ with positive stretching-factor a, where g and $a = |\mathrm{Det} f|^{1/n}$ are uniquely determined by f. The automorphism g is called the v o l u m e - p r e s e r v i n g p a r t and a is called the s t r e t c h i n g - f a c t o r of f.)

†**T9.76** (Tchebychev–Systems) Let K be a field, I be a set and let K^I be the algebra the K–valued functions on I. Further, let $f_1,\ldots,f_n \in K^I$.

(a) The following statements are equivalent :

(i) $f_1,\ldots,f_n$ are linearly independent over K.

(ii) There exist $j_1,\ldots,j_n \in I$ such that the matrix $\mathfrak{V}_{\mathbf{f}}(j_1,\ldots,j_n) := (f_r(j_s))_{1\le r,s\le n} \in \mathrm{M}_n(K)$ is invertible.

(iii) There exist $j_1,\ldots,j_n \in I$ such that the determinant $\mathrm{V}_{\mathbf{f}}(j_1,\ldots,j_n) := \mathrm{Det}\,\mathfrak{V}_{\mathbf{f}}(j_1,\ldots,j_n) \neq 0$.
(**Hint :** See also Test-Exercise T9.?? – The matrices $\mathfrak{V}_{\mathbf{f}}(j_1,\ldots,j_n)$ respectively its determinants $\mathrm{V}_{\mathbf{f}}(j_1,\ldots,j_n)$ corresponding to the system of functions $\mathbf{f} = (f_1,\ldots,f_n)$ are called generalised Vandermonde's matrices respectively determinants and are also called alternants. The usual Vandermonde's matrices and determinants (see Exercise 8.3 and Exercise 9.5-(a)) correspond to the system of the polynomial functions $1,x,\ldots,x^{n-1}$ from K to itself.)

(b) Suppose that $|I| \ge n$. The following statements are equivalent :

(i) For every subset $J \subseteq I$ with $|J| = n$, $f_1 \upharpoonright J,\ldots,f_n \upharpoonright J$ is a basis of K^J.

(ii) For every function $g \in K^I$ and every subset $J \subseteq I$ with $|J| = n$, there exists a unique n–tuple $(b_1,\ldots,b_n) \in K^n$ such that $g(j) = \sum_{r=1}^n b_r f_r(j)$ for all $j \in J$. (iii) For $b_1,\ldots,b_n \in K$, if the function $\sum_{r=1}^n b_r f_r$ has n distinct zeros on I, then $b_1 = \cdots = b_n = 0$.

(iv) For distinct elements $j_1,\ldots,j_n \in I$, the generalised Vandermonde's matrix $\mathfrak{V}_{\mathbf{f}}(j_1,\ldots,j_n)$ is invertible.

(v) For distinct elements $j_1,\ldots,j_n \in I$, the generalised Vandermonde's determinant $\mathrm{V}_{\mathbf{f}}(j_1,\ldots,j_n) \neq 0$. (**Remark:** A system $\mathbf{f} = (f_1,\ldots,f_n)$ of functions in K^I which satisfy these equivalent conditions is called a Tchebychev–System on I.)

(c) Let $(f_1,\ldots,f_n)$ be a Tchebychev–system on I (see Remark in part (b)). Then
(1) $(f_1 \upharpoonright I',\ldots,f_n \upharpoonright I')$ is also a Tchebychev–System on I' for every subset $I' \subseteq I$ with $|I'| \ge n$.
(2) If $g_1,\ldots,g_n : I \to K$ generate the same subspace as that generated by $f_1,\ldots,f_n$ in K^I, then $(g_1,\ldots,g_n)$ is also a Tchebychev–system on I.

(d) Let $f_1,\ldots,f_n$ be a Tchebychev–system on I and let $g \in K^I$. Further, let $j_1,\ldots,j_n \in I$ be distinct elements. For the linear combination f of the $f_1,\ldots,f_n$ with $f(j_s) = g(j_s)$ for $s = 1,\ldots,n$. Show that:

$$\begin{vmatrix} f & g(j_1) & \cdots & g(j_n) \\ f_1 & f_1(j_1) & \cdots & f_1(j_n) \\ \cdots & \cdots & \cdots & \cdots \\ f_n & f_n(j_1) & \cdots & f_n(j_n) \end{vmatrix} = 0,$$

(**Remark :** In this case f is determined by expanding this determinant in terms of the first column. We say that the function f is obtained from the function g by interpolation with the system $f_1,\ldots,f_n$ with (interpolation–)knots $j_1,\ldots,j_n$.)

(e) Let I be a topological space and let $K = \mathbb{R}$. Let $\Delta_n(I)$ denote the set of all those tuples in I^n, which have at least two equal components. Suppose that $(j_1,\ldots,j_n) \in I^n \smallsetminus \Delta_n(I)$ and an odd permutation $\sigma \in \mathfrak{S}_n$ such that $(j_1,\ldots,j_n)$ and $(j_{\sigma 1},\ldots,j_{\sigma n})$ belong to the same connected component of $I^n \smallsetminus \Delta_n(I)$. Then show that there is no Tchebychev–system $(f_1,\ldots,f_n)$ on I, where $f_r : I \to \mathbb{R}$, $r = 1,\ldots,n$ are *continuous* functions.

†**T9.77** **(a)** Let K be a field with at least n elements, $n \in \mathbb{N}^*$. Then the polynomial functions $1,x,\ldots,x^{n-1}$ form a Tchebychev–system on K. (this follows from Test-Exercise T?.??.) More generally: If I is a set and $f : I \to K$ is injective, then the powers $1,f,\ldots,f^{n-1}$ for every $n \le |I|$ form a Tchebychev–system on I.

(b) Let K be a field with at least $2n$ elements, $n \in \mathbb{N}^*$ and $a_1, \ldots, a_n$ distinct elements in K. Then the rational functions $f_1(x) = 1/(x-a_1), \ldots, f_n(x) = 1/(x-a_n)$ form a Tchebychev–system on $K \smallsetminus \{a_1, \ldots, a_n\}$. (**Hint :** Consider $ff_1, \ldots, ff_n$ with $f(x) = (x-a_1)\cdots(x-a_n)$.)

(c) The functions $1, \cos t, \ldots, \cos(n-1)t$, $n \in \mathbb{N}^*$ form a Tchebychev–system on the real interval $[0, \pi]$. (**Remark :** $1, \cos t, \ldots, \cos(n-1)t$ respectively, $1, \cos t, \ldots, \cos^{n-1} t$ generate the same function space, see Question T1.4 in the Test 1.)

(d) The functions $\sin t, \ldots, \sin nt$, $n \in \mathbb{N}^*$ form a Tchebychev–system on the open real interval $(0, \pi)$. (**Remark :** $\sin t, \ldots, \sin nt$ respectively, $\sin t, \sin t \cos t, \ldots, \sin t \cos^{n-1} t$ generate the same function space.)

(e) Let $n \in \mathbb{N}$. The $2n+1$ functions $\exp(\mathrm{i}\nu t)$, $\nu = -n, \ldots, -1, 0, 1, \ldots, n$ form a Tchebychev–system on the half-open real interval $[0, 2\pi)$. Similarly the functions $1, \cos t, \sin t, \ldots, \cos nt, \sin nt$ form a Tchebychev–system on $[0, 2\pi)$. (**Remark :** The given system is also a Tchebychev–system on the unit circle S^1. Does there exists a Tchebychev–system with $2n+2$ *continuous* functions $S^1 \to \mathbb{R}$ on the unit circle? See Test-Exercise T9.76-(e).)

†**T9.78** The space $\mathrm{M}_{m,n}(\mathbb{R})$ of real $(m \times n)$–matrices has a natural topology which is defined by the metric $d((a_{ij}), (b_{ij})) := \mathrm{Max}\{|b_{ij} - a_{ij}| : 1 \le i \le m,\ 1 \le j \le n\}$.

(a) The determinant map $\mathrm{Det} : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is continuous.

(b) The set $\mathrm{GL}_n(\mathbb{R})$ of all invertible matrices in $\mathrm{M}_n(\mathbb{R})$ is open in $\mathrm{M}_n(\mathbb{R})$. (**Remark :** In fact, a *dense* open subset of $\mathrm{M}_n(\mathbb{R})$, see also Exercise 9.7-(b).)

(c) Let $r \in \mathbb{N}$. The set of all matrices of rank $\ge r$ in $\mathrm{M}_{m,n}(\mathbb{R})$ is open in $\mathrm{M}_{m,n}(\mathbb{R})$.

(d) The set of all matrices of maximal rank $= \mathrm{Min}\{m, n\}$ in $\mathrm{M}_{m,n}(\mathbb{R})$ is open in $\mathrm{M}_{m,n}(\mathbb{R})$.

†**T9.79** Let U be an open subset in $\mathbb{R}^n$ and let $g : U \to \mathbb{R}^m$ be a *continuously differentiable* map, i. e. the functions $g_i := p_i g$, $i = 1, \ldots, m$, where $p_i : \mathbb{R}^m \to \mathbb{R}$ are the canonical projections on the components, are partial differentiable with continuous partial derivatives $\partial_j g_i = \partial g_i / \partial t_j$, $1 \le i \le m$, $1 \le j \le n$. For $t \in U$, the matrix

$$\mathfrak{J}(g)(t) := \begin{pmatrix} \partial_1 g_1(t), & \cdots, & \partial_n g_1(t) \\ \vdots & \ddots & \vdots \\ \partial_1 g_m(t), & \cdots, & \partial_n g_m(t) \end{pmatrix}$$

is called the functional– or Jacobian–matrix of g in t, in the case $m = n$ its determinant $\mathrm{J}(g)(t) := |\mathfrak{J}(g)(t)|$ is called the functional– or Jacobian–determinant of g in t.

(a) Suppose that $m = n$. Then the function $t \mapsto \mathrm{J}(g)(t)$ is continuous on U.

(b) Suppose that $m = n$. The subset $\{t \in U \mid \mathfrak{J}(g)(t) \text{ is invertible}\}$ is open in U.

(c) Let $r \in \mathbb{N}$. The subset $\{t \in U \mid \mathrm{Rank}\, \mathfrak{J}(g)(t) \ge r\}$ is open in U.

(d) The subset $\{t \in U \mid \mathfrak{J}(g)(t) \text{ has a maximal rank } \mathrm{Min}\{m, n\}\}$ is open in U. (**Remark :** In this case we say that g is regular at such a point in U.)

†**T9.80** **(a)** The map $t \mapsto (1/t_1^2 + \cdots + t_n^2) \cdot t$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n \smallsetminus \{0\}$, has the functional determinant $-(1/(t_1^2 + \cdots + t_n^2)^n)$. (**Hint :** Use Test-Exercise T9.42.)

(b) (Polar coordinates) For the map $g : t \mapsto (g_1(t), \ldots, g_n(t))$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $n \ge 2$, where

$$
\begin{aligned}
g_1(t) &= t_1 \cos t_n \cdots \cos t_3 \cos t_2\\
g_2(t) &= t_1 \cos t_n \cdots \cos t_3 \sin t_2\\
g_3(t) &= t_1 \cos t_n \cdots \sin t_3\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
g_{n-1}(t) &= t_1 \cos t_n \sin t_{n-1}\\
g_n(t) &= t_1 \sin t_n .
\end{aligned}
$$

Show that: $\mathrm{J}(g)(t) = t_1^{n-1} \cos^{n-2} t_n \cdots \cos t_3$. (**Hint :** Induction on n.)