E0 219 Linear Algebra and Applications / August-December 2011

(ME, MSc. Ph. D. Programmes)

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Lectures : Monday and Wednesday ; 11:30–13:00 **Venue:** CSA, Lecture Hall (Room No. 117)

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1-st Midterm : Saturday, September 17, 2011; 15:00 -17:00 **2-nd Midterm :** Saturday, October 22, 2011; 10:30 -12:30
Final Examination : Tuesday, December 06, 2011, 09:00 -12:00

Evaluation Weightage : Assignments : 20% **Midterms (Two) :** 30% **Final Examination :** 50%

Range of Marks for Grades (Total 100 Marks)						
	Grade S	**Grade A**	**Grade B**	**Grade C**	**Grade D**	**Grade F**
Marks-Range	> 90	76–90	61–75	46–60	35–45	< 35

10. Eigen-values, Characteristic Polynomials, Minimal Polynomials

Submit a solution of **any one** of the $*$-Exercise ONLY

Due Date : Wednesday, 30-11-2011 (Before the Class)

- **Solution of the $**$-Exercise (Exercise 10.8) carries 10 Bonus Points.**
- •• **Solution of the $***$-Exercise (Exercise 10.10)) carries 20 Bonus Points!**
- ••• **Highly recommended $****$-Exercise (Test-Exercise T10.50)) for many applications!**

***10.1** Let $V := \mathbb{K}^{\mathbb{R}}$ and let $T \in \mathbb{R}$ be a positive real number. Let $f_T : V \to V$ be the linear operator defined by $f_T(x)(t) := x(t+T)$ for $x \in V$.

(a) Show that 0 is neither a spectral value nor an eigen value for f_T and the eigen-space of f_T at 1 is $V_{f_T}(1) = V_{\mathrm{per},T} := \{x \in V \mid x \text{ is periodic with period } T\}$.

(b) Let $\mathbb{K} = \mathbb{C}$. Show that every $\lambda \in \mathbb{C}^\times$ is an eigen-value of f_T with eigen function $\exp\left(\frac{\ln(\lambda)}{T}t\right)$, where, if λ is a negative real number then we put $\ln(\lambda) := \ln(|\lambda|) + i\pi$ and the eigen-space of f_T at λ is $\exp\left(\frac{\ln(\lambda)}{T}t\right) V_{\mathrm{per},T}$.

(c) Let $\mathbb{K} = \mathbb{R}$. Show that every positive real number λ is an eigen-value of f_T and the eigen-space of f_T at λ is $\lambda^{t/T} V_{\mathrm{per},T}$.

(d) Let $\mathbb{K} = \mathbb{R}$. The eigen-space of f_T at the eigen-value -1 is called the *half periodic functions* and is usually denoted by $V_{\mathrm{hper},T}$. Show that

(i) Every half periodic function is period with period $2T$.

(ii) $V_{\mathrm{hper},T} = \cos\left(\frac{\pi t}{T}\right) V_{\mathrm{per},T} + \sin\left(\frac{\pi t}{T}\right) V_{\mathrm{per},T}$.

(iii) For a positive real number λ, the eigen-space of f_T at $-\lambda$ is $V_{f_T}(-\lambda) = \lambda^{t/T} V_{\mathrm{hper},T}$.

(e) Eigen-function corresponding to an eigen-value $\lambda \neq 1$ are called *periodic functions of second kind with multiplicator* λ. Show that if λ is a n-th root of unity then every eigen-function of second kind with multiplicator λ is periodic with period nT. (**Remark :** The same assertions (a) to (e) hold for the restriction of v_T to the subspaces $C^k_{\mathbb{K}}(\mathbb{R})$, $k \in \mathbb{N} \cup \{\infty, \omega\}$.)

10.2 Let $\mathfrak{A} \in \mathrm{M}_n(K)$, $n \geq 2$ be a nilpotent matrix.

(a) If $\mathfrak{A}^{n-1} \neq 0$, then there does not exists any matrix $\mathfrak{B} \in \mathrm{M}_n(K)$ with $\mathfrak{B}^2 = \mathfrak{A}$.

(b) The following statements are equivalent: (1) $\mu_{\mathfrak{A}} = \chi_{\mathfrak{A}}\ (= X^n)$. (2) $\mathfrak{A}^{n-1} \neq 0$. (3) Rank $\mathfrak{A} = n-1$. (4) There exists a $\mathfrak{x} \in K^n$ such that $\mathfrak{A}^i\mathfrak{x}$, $i = 0, \ldots, n-1$ is a basis of K^n. (**Hint :** Prove the implication (3) $\Rightarrow$ (2) by induction on n.)

10.3 Let $f : V \to V$ be an operator on the K-vector space V. The following statements are equivalent: (1) f is a homothecy. (2) Every subspace of V is f-invariant. (3) Every vector $\neq 0$ in V is an eigen-vector of f.

10.4 Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $n\times n$-matrices over the field K, assume that one of them is invertible. Then there exists atmost n distinct elements $a\in K$ such that the matrix $a\mathfrak{A}+\mathfrak{B}$ is not invertible. (**Hint :** Suppose that $\mathfrak{A}$ is invertible, then $\operatorname{Det}\mathfrak{A}\neq 0$. Now, since $\operatorname{Det}(a\mathfrak{A}+\mathfrak{B})=\operatorname{Det}(a\mathfrak{E}_n+\mathfrak{B}\mathfrak{A}^{-1})\cdot\operatorname{Det}(\mathfrak{A})=\chi_{-\mathfrak{B}\mathfrak{A}^{-1}}(a)\cdot\operatorname{Det}(\mathfrak{A})$, only for at most n eigen-values a of $-\mathfrak{B}\mathfrak{A}^{-1}$, $\operatorname{Det}(a\mathfrak{A}+\mathfrak{B})=0$.
Now suppose that $\mathfrak{B}$ is invertible, then $a\mathfrak{A}+\mathfrak{B}$ is invertible for $a=0$ and for $a\neq 0$, $a\mathfrak{A}+\mathfrak{B}$ is not invertible only for the n eigen-values of $-\mathfrak{A}\mathfrak{B}^{-1}$, since $\operatorname{Det}(a\mathfrak{A}+\mathfrak{B})=\operatorname{Det}(\mathfrak{A}\mathfrak{B}^{-1}+a^{-1}\mathfrak{E}_n)\cdot\operatorname{Det}(\mathfrak{B})=a\cdot\chi_{-\mathfrak{A}\mathfrak{B}^{-1}}(a^{-1})\cdot\operatorname{Det}(\mathfrak{B})$.)

***10.5** Let $n\in\mathbb{N}$ and let K be a field with $k\cdot 1_K\neq 0$ for all $k=1,\ldots,n$.

(a) An operator f on the n-dimensional K-vector space V is nilpotent if and only if $\operatorname{Tr} f=\operatorname{Tr} f^2=\cdots=\operatorname{Tr} f^n=0$. (**Hint :** If f is nilpotent, then so are f^2, f^3, …, f^n and hence the characteristic polynomials $\chi_{f^i}=X^n$, in particular, $\operatorname{Tr} f^i=0$ for all $i=1,\ldots,n$. Prove the converse by induction on n. Since $\operatorname{Tr}(f^i)=0$ for all $i=1,\ldots,n$, by Cayley-Hamilton Theorem $0=\chi_f(f)=f^n-(\operatorname{Tr}(f))f^{n-1}+\cdots+(-1)^n\operatorname{Det}\operatorname{id}_V$ and hence applying the trace map, we get $0=\operatorname{Tr}(\chi_f(f))=\operatorname{Tr}(f^n)-(\operatorname{Tr}(f))\operatorname{Tr}(f^{n-1})+\cdots+(-1)^n\operatorname{Det}\operatorname{Tr}(\operatorname{id}_V)=(1)^n n\operatorname{Det}(f)$. It follows that $\operatorname{Det} f=0$ and hence f is not injective and $\operatorname{Dim}_K\overline{V}<n=\operatorname{Dim}_K V$, where $\overline{V}:=V/\operatorname{Ker} f$. Now use Test-Exercise T10.24 and apply induction.)

(b) Suppose that $a_1,\ldots,a_n$ are elements in K with

$$\begin{aligned} a_1^1+\cdots+a_n^1&=0\\ \cdots&\cdots\\ a_1^n+\cdots+a_n^n&=0. \end{aligned}$$

Then $a_1=\cdots=a_n=0$. (**Hint :** Let $f:K^n\to K^n$ be the linear map defined by the diagonal matrix $\operatorname{Diag}(a_1,\ldots,a_n)$ (with respect to the standard basis $e_1,\ldots,e_n$ of K^n). Then for every $k=1,\ldots,n$, the matrix of f^k (with respect to the standard basis) is the diagonal matrix $\operatorname{Diag}(a_1^k,\ldots,a_n^k)$ and by hypothesis $\operatorname{Tr}(f)=\operatorname{Tr}(f^2)=\cdots=\operatorname{Tr}(f^n)=0$. Now apply the part (a) above, to conclude that $\mathfrak{A}$ is nilpotent. – **Remark:** The parts (a) and (b) are equivalent: There exists (by *Kronecker's Theorem*[1]) a field extension $K\subseteq L$ such that the characteristic polynomial χ_f of f splits into liner factors $\chi_f=(X-a_1)\cdots(X-a_n)$ in $L[X]$. Then the trace $\operatorname{Tr}(f^k)=a_1^k+\cdots+a_n^k$, see Example 11.B.13.)

10.6 Find the characteristic polynomial of the following matrices :

(a) $$\mathfrak{A}:=\begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & b_1\\ 0 & a_2 & \cdots & 0 & 0 & \cdots & b_2 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & ⋰ & \vdots & \vdots\\ 0 & 0 & \cdots & a_n & b_n & \cdots & 0 & 0\\ 0 & 0 & \cdots & b_n & a_n & \cdots & 0 & 0\\ \vdots & \vdots & ⋰ & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & b_2 & \cdots & 0 & 0 & \cdots & a_2 & 0\\ b_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix}\in\mathrm{M}_{2n}(K).$$

(**Ans :** $\chi_{\mathfrak{A}}=\prod_{k=1}^n(X-a_k-b_k)(X-a_k+b_k)$.) (**Hint :** See Test-Exercise T9.55-(c).)

(b) $$\mathfrak{A}:=\begin{pmatrix} a & b_2 & \cdots & b_n\\ c_2 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ c_n & 0 & \cdots & 0 \end{pmatrix}\in\mathrm{M}_n(K).$$ (**Ans :** $\chi_{\mathfrak{A}}=X^n-aX^{n-1}-\left(\sum_{k=2}^n b_kc_k\right)X^{n-2}\ \ n\geq 2$.)

[1]**Kronecker's Theorem** *Let K be a field and let $P\in K[X]$ be a non-zero polynomial. Then there exists a field extension $K\subseteq L$ such that P factores into linear factors in $L[X]$. Moreover, one can also choose L such that L has finite dimension over K (as an K-algebra).*

(c) $$\mathfrak{F}_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{M}_n(\mathbb{R}).$$

(**Ans :** $2^n U_n(X/2)$, where U_n is the n-th Tchebychev polynomial of second kind (see Test-Exercise T9.53-(c)) In particular, $\lambda_k := 2\cos(k\pi/(n+1))$, $k = 1,\dots,n$ are eigen-value s of $\mathfrak{F}_n$. The vector with components $\sin(k\pi i/(n+1))$, $i = 1,\dots,n$ is an eigen-vector corresponding to λ_k.)

10.7 Let f and g be operators on the K-vector space V.

(a) If either fg or gf is algebraic, then both fg and gf are algebraic and the minimal polynomials of fg and gf are either equal or differ by the factor X. Moreover, if either f or g is invertible, then $\mu_{fg} = \mu_{gf}$. Give examples of operators f and g on K^2 such that $\mu_{fg} \neq \mu_{gf}$.

(b) Suppose that V is finite dimensional. Then $\chi_{fg} = \chi_{gf}$. (**Hint :** Use Exercise 8.4-(b) to assume that either f is invertible or f is a projection.)

****10.8** Let f be an operator on the K-vector space V and let $x \in V$. Then show that

(a) $V_x := \sum_{m\in\mathbb{N}} K f^m(x)$ is the smallest f-invariant subspace of V which contain x. (**Remark :** The subspace V_x is called the f-cyclic subspace generated by x.)

(b) V_x is finite dimensional if and only if there exists a monic polynomial $P \in K[X]$ such that $P(f)(x) = 0$. Moreover, in this case, if P_x is the monic polynomial of the smallest degree with $P_x(f)(x) = 0$, then P_x is the minimal polynomial and the characteristic polynomial of $f|V_x$. (**Remark :** This polynomial P_x is called the f-annihilator of x and denoted by $\mathrm{ann}_f(x)$. With this $\mathrm{Deg}\,\mathrm{ann}_f(x) = \mathrm{Dim}_K V_x$.)

(c) If V is finite dimensional and $x_1,\dots,x_r$ is a generating system for V, then μ_f is equal to $\mathrm{LCM}(P_{x_1},\dots,P_{x_r})$. (**Hint :** It is enough to prove the equality $V = \sum_{\rho=1}^r V_{x_\rho}$, see Test-Exercise T10.14.)

(d) Suppose that V is finite dimensional. Then the following statements are equivalent :

(i) $V_{x_0} = V$ for some $x_0 \in V$.

(ii) There exists a K-basis $\mathfrak{v} = \{v_1,\dots,v_n\}$ of V such that the matrix of f with respect to the basis $\mathfrak{v}$ is of the form

$$\mathfrak{A}_P := \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in \mathrm{M}_n(K)$$

(iii) $\chi_f = \mu_f$.

(**Remark :** If any one of the above equivalent statements hold, then the operator f is called a cyclic operator and the element x_0 is called a cyclic element for f. The matrix $\mathfrak{A}_P$ is called the companion matrix of the polynomial P.)

(e) If χ_f has only *simple prime factors*, then f is cyclic. (**Hint :** In this case $\chi_f = \mu_f$ by 11.A.14.)

10.9 Let V be a finite dimensional K-vector space of dimension n.

(a) Let f and g be invertible operators on V. Then all operators $\lambda f - \mu g$, $(\lambda,\mu) \in K^2 - \{(0,0)\}$ are invertible if and only if the characteristic polynomial $\chi_{f^{-1}g}$ of $f^{-1}g$ has no zeroes, i.e. $f^{-1}g$ has no eigen-value.

(b) Let $\Phi\colon V\times V\to V$ be bilinear. If K is algebraically closed and $n\geq 2$, then Φ has a zero divisor, i.e. there exist $x,y\in V$ with $x\neq 0\neq y$ and $\Phi(x,y)=0$. If $K=\mathbb{R}$ and n is odd and ≥ 3, then Φ has a zero divisor. (**Hint :** For $x\in V$ consider the operators $f_x\colon y\mapsto \Phi(x,y)$ on V. – A deep theorem states that *if $K=\mathbb{R}$ and $n\neq 0,1,2,4,8$, then Φ has a zero divisor*.)

******10.10*** Let $\lambda\in\mathbb{K}$ be an eigen-value of the matrix $\mathfrak{A}=(a_{ij})\in\mathrm{M}_n(\mathbb{K})$. Then $|\lambda-a_{ii}|\leq\sum_{j\neq i}|a_{ij}|$ for at least one $i\in\{1,\ldots,n\}$ and also $|\lambda-a_{jj}|\leq\sum_{i\neq j}|a_{ij}|$ for at least one $j\in\{1,\ldots,n\}$.

(a) In particular, (Gershgorin circle theorem[2]): *the (eigen) spectrum* $\operatorname{Spec}\mathfrak{A}$ *is contained in the union* $\cup_{i=1}^{n}\mathrm{D}_1(\mathfrak{A})$ *of the closed discs* $\mathrm{D}_i(\mathfrak{A}):=\overline{\mathrm{B}}(a_{ii},R_i)$ *centered at* a_{ii} *and radius* $R_i:=\sum_{j\neq i}|a_{ij}|$, $i=1,\ldots,n$. The closed discs $\mathrm{D}_i(\mathfrak{A})$, $i=1,\ldots,n$, are called the Gershgorin discs. – For a diagonal matrix $\mathfrak{D}$, the union of the Gershgorin discs $\cup_{i=1}^{n}\mathrm{D}_i(\mathfrak{D})$ coincides with the spectrum $\operatorname{Spec}\mathfrak{D}$, and conversely. (Use Exercise 4.2, see also Exercise 9.7. – **Remark:** The Gershgorin circle theorem is useful in solving matrix equations of the form $\mathfrak{A}\mathfrak{x}=\mathfrak{b}$ for $\mathfrak{x}$, where $\mathfrak{b}$ is a vector and $\mathfrak{A}$ is a matrix with a large *condition number*.)

(b) In general Gershgorin circle theorem in the part (a) can be strengthened as follows: *If the union* $\mathrm{D}(\mathfrak{A}):=\mathrm{D}_{i_1}\cup\cdots\cup\mathrm{D}_{i_k}$ *of k Gershgorin-discs is disjoint from the union* $\mathrm{D}'(\mathfrak{A}):=\cup_{i\in\{1,\ldots,n\}\setminus\{i_1,\ldots,i_k\}}\mathrm{D}_i$ *of the other $n-k$ Gershgorin-discs then* $\mathrm{D}(\mathfrak{A})$ *contains exactly k and* $\mathrm{D}'(\mathfrak{A})$ *$n-k$ eigen-values of* $\mathfrak{A}$. (**Hint:** The assertion is obviously true for diagonal matrices. For a proof consider $\mathfrak{B}(t):=(1-t)\mathfrak{D}+t\mathfrak{A}$, $t\in[0,1]$, where $\mathfrak{D}:=\mathrm{Diag}(a_{11},\ldots,a_{nn})$. Note that the hypothesis $\mathrm{D}(\mathfrak{A})\cap\mathrm{D}'(\mathfrak{A})=\emptyset$, yields $\mathrm{D}(\mathfrak{B}(t))\cap\mathrm{D}'(\mathfrak{B}(t))=\emptyset$ for all $t\geq 0$, since the centers of the Gershgorin discs of $\mathfrak{B}(t)$ are same as those of $\mathfrak{A}$ and the radii are t times those of $\mathfrak{A}$. Let $d(t):=d(\mathrm{D}(\mathfrak{B}(t)),\mathrm{D}'(\mathfrak{B}(t)))$ denote the distance between $\mathrm{D}(\mathfrak{B}(t))$ and $\mathrm{D}'(\mathfrak{B}(t))$. Then $d(0)=d(\mathfrak{D})\geq d(t)\geq d(\mathfrak{A})=d(1)>0$ (since the discs are closed and the function $t\mapsto d(t)$ is decreasing). Since the eigen-values of $\mathfrak{B}(t)$ are continuous functions of t (this is proved below), for any eigen-value $\lambda(t)$ of $\mathfrak{B}(t)$ in $\mathrm{D}(\mathfrak{B}(t))$, its distance $\delta(t):=d(\lambda(t),\mathrm{D}'(t))$ is also continuous. Obviously $\delta(t)\geq d(t)\geq d(1)>0$ for all $t\in[0,1]$ and in particular, $\delta(0)\geq d(1)>0$. Note that since the assertion is obviously true for the diagonal matrices, there are exactly k eigen-values $\lambda_1(0),\ldots,\lambda_k(0)$ of $\mathfrak{D}$ in $\mathrm{D}(\mathfrak{D})$. We shall use this and the continuity of the function ,δ to show that the eigen-values $\lambda_1(1),\ldots,\lambda_k(1)$ of $\mathfrak{A}$ are in $\mathrm{D}(\mathfrak{D})$. For this we fix $i\in\{1,\ldots,k\}$ and put $\lambda(t):=\lambda_i(t)$. Suppose on the contrary that $\lambda(1)\in\mathrm{D}'(\mathfrak{A})=\mathrm{D}'(\mathfrak{B}(1))$. Then $\delta(1)=0$, and hence $\delta(0)\geq d(0)>d(1)>0=\delta(1)$. Therefore by *Intermediate value Theorem* (see Footnote 4 on Page 4 of Exercise Set 9) there exists a $t_0\in(0,1)$ such that $\delta(t_0)=d(1)$. But, then $\delta(t_0)=d(1)<d(t_0)\leq\delta(t_0)$, which is impossible. This proves the assertion.

Now we shall indicate the proof of the assertion: *The zeros of a monic complex polynomial are continuous functions of its coefficients*, which is used in the above proof. More precisely:

Lemma *Let λ be a zero of the polynomial $X^n+a_{n-1}X^{n-1}+\cdots+a_0\in\mathbb{C}[X]$ of multiplicity m. Further, let $\varepsilon>0$ be given. Then there exists a $\delta>0$ such that all polynomials $X^n+b_{n-1}X^{n-1}+\cdots+b_0\in\mathbb{C}[X]$ with $|b_i-a_i|\leq\delta$ for $i=0,\ldots,n-1$ have at least m, zeroes in the (open) disc $\mathrm{B}(\lambda\,;\varepsilon)$, every zero is counted with its multiplicity.*

Proof. We consider the continuous map $\Phi\colon\mathbb{C}^n\to\mathbb{C}^n$, which maps every n-tuple of complex numbers $(\lambda_1,\ldots,\lambda_n)$ to the n-tuple $(a_0,\ldots,a_{n-1})$ of the coefficients (other than the leading coefficient) of the polynomial $(X-\lambda_1)\cdots(X-\lambda_n)$. Then Φ is surjective by the *Fundamental Theorem of Algebra*[3], and the fibre of Φ passing through the n-tuple $(\lambda_1,\ldots,\lambda_n)$ is the set of all n-tuples $\sigma(\lambda_1,\ldots,\lambda_n)=(\lambda_{\sigma^{-1}1},\ldots,\lambda_{\sigma^{-1}n})$, $\sigma\in\mathfrak{S}_n$. Further, *if $A\subseteq\mathbb{C}^n$ is a closed subset, then its image $\Phi(A)$ is also closed subset.* For, if $\Phi(x_\nu)$,

[2] It was first published by the Belarusian mathematician Semyon Aranovich Gershgorin (1901-1933) in 1931, see [**Gerschgorin, S.** Über die Abgrenzung der Eigenwerte einer Matrix, *Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk*, **7** (1931), 749-754]. He studied at Petrograd Technological Institute from 1923, becoming Professor in 1930, and from 1930 he worked in the Leningrad Mechanical Engineering Institute on algebra, theory of functions of complex variables, numerical methods and differential equations.

[3] **Fundamental Theorem of Algebra** (d'Alembert - Gauss) *Every non-constant polynomial $f\in\mathbb{C}[X]$ has a zero in $\mathbb{C}$.* – Jean d'Alembert (1717-1783) was a a French mathematician who was a pioneer in the study of differential equations and their use of in physics. He studied the equilibrium and motion of fluids. – Johann Carl Friedrich Gauss (1777-1855) was a German mathematician who worked in a wide variety of fields in both mathematics and physics including number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. His work has had an immense influence in many areas.

$\nu\in\mathbb{N}$, $x_\nu\in A$, is a convergent sequence in $\Phi(A)$, then $x_\nu\in A$, is a bounded sequence by the Exercise[4] and hence by the *Bolzano-Weierstrass Theorem*[5] x_ν, $\nu\in\mathbb{N}$, has a convergent subsequence. We may therefore assume that x_ν, $\nu\in\mathbb{N}$, is already convergent. Then, if $x:=\lim x_\nu\in A$, then $\Phi(x)=\lim\Phi(x_\nu)\in\Phi(A)$. Therefore it follows that: *If $U\subseteq\mathbb{C}^n$ open, then its image $\Phi(U)$ is also open.* The complement of $\Phi(U)$ in $\mathbb{C}^n$ is $\Phi\big(\mathbb{C}^n-\bigcup_{\sigma\in\mathfrak{S}_n}\sigma(U)\big)$ and hence it is closed by the above proof.

Let $X^n+a_{n-1}X^{n-1}+\cdots+a_0=(X-\lambda_1)\cdots(X-\lambda_n)$ and $\varepsilon>0$ be given. Then $\Phi\big(\mathrm{B}(\lambda_1;\varepsilon)\times\cdots\times\mathrm{B}(\lambda_n;\varepsilon)\big)$ is an open neighbourhood of $(a_0,\ldots,a_{n-1})$, which contains a product $\overline{\mathrm{B}}(a_0;\delta)\times\cdots\times\overline{\mathrm{B}}(a_{n-1};\delta)$ of discs with $\delta>0$. This proves the assertion. •)

Below one can see auxiliary results and (simple) Test-Exercises.

Auxiliary Results/Test-Exercises

To understand and appreciate the Test-Exercises which are marked with the symbol † one may possibly require more mathematical maturity than one has! These are steps towards applications to various other branches of mathematics, especially to Analysis, Number Theory, Graph Theory, Group Theory and Affine and Projective Geometry.

T10.1 Let $n\in\mathbb{N}^+$ and let $V:=\mathbb{K}[t]_n$. For the linear operators $D:=d/dt:V\to V$ defined by $P\mapsto P':=d/dt(P)$ and $f:V\to V$ defined by $P\mapsto P(t+1)$ compute the characteristic polynomial, minimal polynomial, eigen-values and eigen-spaces. (**Ans:** $\chi_D=X^n=\mu_D$ and $\chi_f=(X-1)^n=\mu_f$. – **Hint:** The matrix $\mathfrak{A}=\mathfrak{M}^{\mathfrak{t}}_{\mathfrak{t}}(D)$ (respectively $\mathfrak{B}=\mathfrak{M}^{\mathfrak{t}}_{\mathfrak{t}}(f)$) of the operator D (respectively f) with respect to the basis $\mathfrak{t}:=(1,t,\ldots,t^{n-1})$ of $V=K[t]_n$ are

$$\mathfrak{A}:=\begin{pmatrix}0&1&0&\cdots&0&0&0&\cdots&0&0\\0&0&2&\cdots&0&0&0&\cdots&0&0\\0&0&0&\cdots&0&0&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&0&i&0&\cdots&0&0\\0&0&0&\cdots&0&0&i+1&\cdots&0&0\\0&0&0&\cdots&0&0&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&0&0&0&\cdots&0&n-1\\0&0&0&\cdots&0&0&0&\cdots&0&0\end{pmatrix}\quad\text{and}\quad\mathfrak{B}:=\begin{pmatrix}1&1&1&\cdots&1&1&\cdots&1\\0&1&2&\cdots&j-1&j&\cdots&n-1\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\binom{i}{i-1}&\binom{i+1}{i-1}&\cdots&0\\0&0&0&\cdots&1&\binom{i+1}{i}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0&0&\cdots&n-1\\0&0&0&\cdots&0&0&\cdots&1\end{pmatrix}$$

Therefore $\chi_D=\mathrm{Det}\,(X\mathfrak{E}-\mathfrak{A})=X^n$ and e-Spec $(D)=\mathrm{Z}_{\mathbb{K}}(\chi_D)=\{0\}$. Further, since $\deg P'=\deg P-1$ for every non-constant $P\in\mathbb{K}[t]_n$. It follows that the eigen-space $V_D(0)=\mathrm{Ker}\,D=\mathbb{K}$ (=the space of constant polynomials) and since $D^{n-1}(t^{n-1})=(n-1)!\neq0$. Therefore $D^{n-1}\neq0$ and hence $\mu_D=X^n=\chi_D$, since μ_D divides χ_D.

Further, $\chi_f=\mathrm{Det}\,(X\mathfrak{E}-\mathfrak{B})=(X-1)^n$, e-Spec $(D)=\mathrm{Z}_{\mathbb{K}}(\chi_f)=\{1\}$ and since $(t+1)^j-t^j=jt^{j-1}+\cdots$, we have $\deg(f-\mathrm{id})(P)=\deg(P(t+1)-P(t))=\deg P(t)-1$ for every non-constant $P\in\mathbb{K}[t]_n$. It follows that the eigen-space $V_f(1)=\mathrm{Ker}\,(f-\mathrm{id})=\mathbb{K}$ (=the space of constant polynomials) and since $(f-\mathrm{id})^{n-1}(t^{n-1})=(n-1)!\neq0$. Therefore $(f-\mathrm{id})^{n-1}\neq0$ and hence $\mu_f=(X-1)^n=\chi_f$, since μ_f divides χ_f.)

T10.2 Let D be the differentiation operator $f\mapsto f'$ on the vector space $\mathrm{C}^\infty_{\mathbb{K}}(\mathbb{R})$ of infinitely many times differentiable $\mathbb{K}$-valued functions on $\mathbb{R}$. Compute the eigen-values, spectral-values and eigen-spaces for D. (**Ans:** e-Spec $(D)=\mathrm{Spec}\,D=\mathbb{K}$ and $\mathrm{V}_D(\lambda)=\mathbb{K}e^{\lambda x}$ is the eigen-space of $\lambda\in\mathbb{K}$.)

[4] **Exercise** Let $f=a_0+a_1x+\cdots+a_{n-1}x^{n-1}+x^n$ be a monic polynomial in $\mathbb{C}[X]$. Then for every zero α of f in $\mathbb{C}$ prove the estimates: **(a)** $|\alpha|\le\mathrm{Max}\,(1,|a_0|+\cdots+|a_{n-1}|)$. **(b)** $|\alpha|\le\mathrm{Max}\,(|a_0|,1+|a_1|,\ldots,1+|a_{n-1}|)$. **(c)** (Cauchy's Estimates) $|\alpha|\le 2R$ mit $R:=\mathrm{Max}\,(|a_\nu|^{1/(n-\nu)},\nu=0,\ldots,n-1)$. (**Hint :** From $|\alpha|>2R$ and $f(\alpha)=0$, we get $|\alpha|^n=|a_0+\cdots+a_{n-1}\alpha^{n-1}|\le\sum_{\nu=0}^{n-1}R^{n-\nu}|\alpha|^\nu=R\big(|\alpha|^n-R^n\big)/\big(|\alpha|-R\big)<|\alpha|^n$, a contradiction.)

[5] **Theorem** (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a limit point.*

T10.3 Let $P = X^n + a_{n-1}X^{n-1} + \cdots a_1X + a_0 = (X-\lambda_1)^{r_1}\cdots(X-\lambda_m)^{r_m}$ be a monic polynomial with coefficients $a_0,\ldots,a_{n-1}\in\mathbb{C}$ and pairwise distinct zeros $\lambda_1,\ldots,\lambda_m\in\mathbb{C}$ of multiplicities $r_1,\ldots,r_m>0$, respectively. Let $V := \{y\in \mathrm{C}^n_{\mathbb{C}}(\mathbb{C}) \mid P(D)y=0\}$ be the $\mathbb{C}$-vector space of the complex-valued solutions of the homogeneous linear differential equation of n-th order $P(D)y = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots a_1y' + a_0y = 0$. Show that the differentiation $D: V\to V$, $y\mapsto Dy = y'$ is a $\mathbb{C}$-linear operator on V and compute its minimal polynomial, characteristic polynomials, e-Spec D and the eigen-spaces. (**Hint :** By construction $V = \mathrm{Ker}\,P(D)$ and $\mathrm{Dim}_{\mathbb{C}}V = r_1+\cdots+r_m == n = \deg P$. Since $P(D)y=0$, it follows that $P(D)(Dy) = D(P(D)y) = 0$ and hence D induces an operator on V. Further, since $P(D)=0$ on V, the minimal polynomial μ_D divides P by the definition of minimal polynomial. Since $V\subseteq \mathrm{Ker}\,\mu_D(D)$, it follows that $\deg P = \mathrm{Dim}_{\mathbb{C}}V \leq \mathrm{Dim}_{\mathbb{C}}\mathrm{Ker}\,\mu_D(D) = \deg\mu_D$ and hence $\mu_D = P$. Moreover by Cayley-Hamilton Theorem $\chi_D = \mu_D = P$. The eigen-spectrum e-Spec$(D) = \mathrm{Z}(\chi_D) = \{\lambda_1,\ldots,\lambda_m\}$ and the corresponding eigen-spaces $\mathrm{V}_D(\lambda_i) = \mathrm{Ker}\,(\lambda_i\mathrm{id}-D) = \mathbb{C}e^{\lambda_i t}$, $i=1,\ldots,m$, since $y\in\mathrm{Ker}\,(\lambda_i\mathrm{id}-D)$ if and only if y is a solution of the differential equation $y'-\lambda_i y = 0$.)

T10.4 For $k\in\mathbb{N}\cup\{\infty\}$, let S denote the integration operator $f\longmapsto \big(t\mapsto\int_0^t f(\tau)\,d\tau\big)$ on the vector space $\mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$ of the k-times continuously differentiable $\mathbb{K}$-valued functions on $\mathbb{R}$. Then S has no eigen-value and 0 is the only spectral value S, i. e. e-Spec$(S)=\emptyset$ and Spec $S=\{0\}$. (**Hint:** From $S(f)=0$, $f\in\mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$, it follows that $f=0$ by differentiating with respect to the upper limit of the integral. Therefore S is injective and hence 0 is not an eigen-value of S. The operator S is not surjective, since from $S(f)=g$, $f,g\in\mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$, it follows that $g(0)=\int_0^0 f(\tau)d\tau = 0$, and hence no g with $g(0)\neq 0$ can belong to Im(S)

Now, let $\lambda\neq 0$. We shall show that λ is not an eigen-value of S, i. e. $S-\lambda\mathrm{id}$ is injective: For an $f\in\mathrm{C}^1_{\mathbb{K}}(\mathbb{R})$ with $S(f)-\lambda f = 0$ implies that $f(0) = \lambda^{-1}S(f)(0) = \lambda^{-1}\int_0^0 f(\tau)d\tau = 0$ and $\int_0^t f(\tau)d\tau - \lambda f = 0$. Differentiating we get $f-\lambda f' = 0$, i. e. $f' = \lambda^{-1}f = 0$. Solutions of this differential equation have the form $f(t) = ce^{\lambda^{-1}t}$ with a constant $c\in\mathbb{K}$. But, since $f(0)=0$, $c=0$ and hence $f=0$. This proves that

To show that $\lambda\neq 0$ cannot be a spectral value of S, i. e. $S-\lambda\mathrm{id}$ is surjective. Therefore for $g\in\mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$, we need to construct a function $f\in\mathrm{C}^k_{\mathbb{K}}(\mathbb{R})$ such that $(S-\lambda\mathrm{id})(f) = g$. In the case $k\geq 1$, i. e. if g is continuously differentiable, this is easy: The (unique) solution f of the linear differential equation $y' = \lambda^{-1}y - \lambda^{-1}g$ with $f(0) = -\lambda^{-1}g(0)$ is a C^k-function and is the inverse image of g under $(S-\lambda\mathrm{id})$, since $(S-\lambda\mathrm{id})(f) = \int_0^t f(\tau)d\tau - \lambda f(t) = \int_0^t(\lambda f(\tau)+g'(\tau))d\tau - \lambda f(t) = (\lambda f(t)+g(t)) + (\lambda f(0)+g(0)) - \lambda f(t) = g(t)$. The case $k=0$, i. e. g is only continuous, is more difficult and use more analysis!.)

T10.5 Show that the characteristic polynomial of the diagonal matrix $\mathfrak{D} = \mathrm{Diag}(a_1,\ldots,a_n)\in\mathrm{M}_n(K)$ is $\chi_{\mathfrak{D}} = \prod_{i=1}^n = (X-a_i)$ and the minimal polynomial $\mu_{\mathfrak{D}} = \prod_\rho^r(X-a_{i_\rho})$, where $a_{i_1},\ldots,a_{i_r}$ are the distinct elements among $a_1,\ldots,a_n$. Further, show that $\mathfrak{D}$ is cyclic (see Exercise 10.8-(d))if and only if $a_1,\ldots,a_n$ are distinct. Moreover, in this case $x_1+\cdots+x_n$ is a cyclic vector (see Exercise 10.8-(d)) for every operator $f: V\to V$ whose matrix with respect to a basis $x_1,\ldots,x_n$ of a K-vector space V is $\mathfrak{D}$.

T10.6 Let $\mathfrak{E}_\sigma\in\mathrm{M}_n(K)$ be the matrix of the permutation $\sigma\in\mathfrak{S}-n$, i. e. $\mathfrak{E}_\sigma = \big(\delta_{i\sigma(j)}\big)$. In the canonical cycle decomposition of σ, suppose that m_i cycles of order i for $i=1,\ldots,n$. Then $n=\sum_{i=1}^n i\cdot m_i$ and show that the characteristic polynomial and the minimal polynomial of $\mathfrak{E}_\sigma$ are, respectively:

$$\chi_{\mathfrak{E}_\sigma} = \prod_{i=1}^n (X^i-1)^{m_i} \quad\text{and}\quad \mu_{\mathfrak{E}_\sigma} = \mathrm{LCM}\,(X^{i_1}-1,\ldots,X^{i_r}-1)\,,$$

where $i_1,\ldots,i_r$ are the indices i with $m_i\neq 0$. Moreover, $\mathfrak{E}_\sigma$ is cyclic (see Exercise 10.8-(d)) if and only if σ is a cycle of order n. (**Hint :** See also Test-Exercise T10.14.)

T10.7 Let f be an operator on the n-dimensional K-vector space V. Suppose that the degree of the minimal polynomial μ_f is m. Then show that

(a) $\chi_{f+a\mathrm{id}}(X) = \chi_f(X-a)$ and $\mu_{f+a\mathrm{id}}(X) = \mu_f(X-a)$, $a\in K$.

(b) $\chi_{af}(X) = a^n\chi_f(X/a)$ and $\mu_{af}(X) = a^m\,\mu_f(X/a)$, $a\in K^\times$.

(c) If f is invertible, then $\chi_{f^{-1}}(X)=\dfrac{(-1)^n}{\operatorname{Det} f}X^n\chi_f(1/X)$ and $\mu_{f^{-1}}(X)=\dfrac{1}{\mu_f(0)}X^m\mu_f(1/X)$, Further, deduce that : $f^{-1}=\dfrac{\mu-\mu(0)}{\mu(0)X}(f)$ and that eigen-value s of f are all non-zero and $0\neq\lambda\in K$ is an eigen-value of f if and only if λ^{-1} is an eigen-value of f^{-1}.

T10.8 Let V be a K-vector space and let $f:V\to V$ be a linear operator. Show that

(a) f is a projection if and only if the minimal polynomial of f is a divisor of $X(X-1)=X^2-X$.

(b) f is an involution if and only if the minimal polynomial of f is a divisor of $(X+1)(X-1)=X^2-1$.

(c) For a projection (respectively involution) on a finite dimensional vector space find the characteristic polynomial. (**Hint :** in the case of the involution give special attention to the case $1+1=0$ in K, i. e. $\operatorname{Char}K=2$. – **Ans:** $\chi_f=(X-a)^r\cdot X^r$ with $r=\operatorname{Rank} f$; in particular, $\operatorname{Tr} f=\operatorname{Rank} f$ (respectively $\chi_f=(X+1)^r(X-1)^{n-r}$ if $\operatorname{Char}K\neq 2$ (since $\frac{1}{2}(\mathrm{id}_V-f)$ is a projection) and $\chi_f=(X-1)^n$ if $\operatorname{Char}K=2$.)

T10.9 Let $f:V\to V$ be an operator of rank r on the n-dimensional K-vector space V.

(a) χ_f is divisible by X^{n-r}. **(b)** μ_f has degree $\leq r+1$. (**Hint:** Note that $\operatorname{Ker} f$ is an f-invariant subspace of f of dimension $n-r$ by the Rank-Theorem, $f\mid\operatorname{Ker} f=0$ and hence $\chi_{f|\operatorname{Ker} f}=X^{n-r}$, $\mu_{f|\operatorname{Ker} f}=X$ and $\deg\mu_{\overline{f}}\leq\deg\chi_{\overline{f}}=\operatorname{Dim}_K\overline{V}=\operatorname{Dim}_K V-\operatorname{Dim}_K\operatorname{Ker} f=\operatorname{Rank} f=r$, where $\overline{f}:\overline{V}\to\overline{V}$ is the operator induced by f on the quotient space $\overline{V}:=V/\operatorname{Ker} f$. Therefore by 11.A.8 $\chi_f=\chi_{f|\operatorname{Ker} f}\cdot\chi_{\overline{f}}=X^{n-r}\cdot\chi_{\overline{f}}$ and μ_f divides $\mu_{f|\operatorname{Ker} f}\cdot\mu_{\overline{f}}=X\cdot\mu_{\overline{f}}$, in particular, χ_f is divisible by X^{n-r} and $\deg\mu_f\leq r+1$. See also Test-Exercise T10.15.)

T10.10 **(a)** The characteristic polynomial of the $n\times n$-matrix

$$\mathfrak{A}=\begin{pmatrix} a & b & \cdots & b\\ b & a & \cdots & b\\ \vdots & \vdots & \ddots & \vdots\\ b & b & \cdots & a\end{pmatrix}$$

is $(X+b-a)^{n-1}(X-a-(n-1)b)$. Compute its minimal polynomial. determine the conditions on a and b so that $\mathfrak{A}$ is invertible, moreover, in these cases, compute the inverse of this matrix. (**Hint :** See also Test-Exercise T9.52-(a).)

(b) Let $\mathfrak{A}_0,\ldots,\mathfrak{A}_{n-1}\in\mathrm{M}_m(K)$. The characteristic polynomial of the $mn\times mn$-matrix

$$\mathfrak{B}:=\begin{pmatrix} 0 & 0 & \cdots & 0 & -\mathfrak{A}_0\\ \mathfrak{E}_m & 0 & \cdots & 0 & -\mathfrak{A}_1\\ 0 & \mathfrak{E}_m & \cdots & 0 & -\mathfrak{A}_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \mathfrak{E}_m & -\mathfrak{A}_{n-1}\end{pmatrix}$$

is $\operatorname{Det}(X^n\mathfrak{E}_m+X^{n-1}\mathfrak{A}_{n-1}+\cdots+X\mathfrak{A}_1+\mathfrak{A}_0)$.

(c) Let $\mathfrak{A}=\operatorname{Diag}(a_1,\ldots,a_n)\in\mathrm{M}_n(K)$ be a diagonal matrix and let $\mathfrak{B}=(b_{ij})\in\mathrm{M}_n(K)$ be a matrix of rank ≤ 1. Then

$$\chi_{\mathfrak{A}-\mathfrak{B}}=\prod_{i=1}^{n}(X-a_i)+\sum_{j=1}^{n}b_{jj}\prod_{i\neq j}(X-a_i).$$

If $\mathfrak{A}$ is invertible, then $\mathfrak{A}-\mathfrak{B}$ is invertible if and only if $c:=\sum_{j=1}^{n}b_{jj}a_j^{-1}\neq 1$. Further, in this case

$$(\mathfrak{A}-\mathfrak{B})^{-1}=\frac{1}{1-c}\big((1-c)\,a_i^{-1}\delta_{ij}+a_i^{-1}b_{ij}a_j^{-1}\big)_{1\leq i,j\leq n}.$$

T10.11 Let f be a linear operator on the K-vector space V. In the parts (c) and (d) below assume that $\mathrm{Dim}_K V = n \in \mathbb{N}$. Show that

(a) f is nilpotent if and only if μ_f is a power of X. Deduce that: if f is nilpotent, then $\mathrm{Tr}(f) = 0$ and $\mathrm{Det}\,(f) = 0$.

(b) f is unipotent, i. e. $f - \mathrm{id}$ is nilpotent if and only if μ_f is a power of $X-1$. Deduce that: if f is nilpotent, then $\mathrm{Tr}(f) = n$ and $\mathrm{Det}\,(f) = 1$.

(c) f is nilpotent if and only if $\chi_f = X^n$. (**Hint :** Use Cayley-Hamilton Theorem.)

(d) f is unipotent if and only if $\chi_f = (X-1)^n$.

T10.12 Let $K \subseteq L$ be a field extension and let $\mathfrak{A} \in \mathrm{M}_n(K) \subseteq \mathrm{M}_n(L)$. For the minimal- as well as the characteristic polynomial of $\mathfrak{A}$ are independent if the matrix $\mathfrak{A}$ is considered over K or over L. (**Hint :** For the minimal polynomial use the Test-Exercise T9.3.)

T10.13 Let f and g be two commuting operators on the K-vector space V and assume that the operator g is nilpotent. Then $\chi_{f+g} = \chi_f$ and in particular, $\mathrm{Det}\,(f+g) = \mathrm{Det}\, f$ and $\mathrm{Tr}\,(f+g) = \mathrm{Tr}\, f$. (**Hint :** It is enough to prove the assertion for matrices. Further, $X\mathfrak{E}_I - (\mathfrak{A}+\mathfrak{B}) = (X\mathfrak{E}_I - \mathfrak{A})\big(\mathfrak{E}_I - (X\mathfrak{E}_I - \mathfrak{A})^{-1}\mathfrak{B}\big)$, if $\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A}$ and if $\mathfrak{B}$ is nilpotent. – **Remark:** Note that the matrix $X\mathfrak{E}_I - \mathfrak{A}$ is invertible in $\mathrm{M}_I(K(X))$.)

T10.14 Suppose that the K-vector space V is the sum of invariant subspaces U and W under the K-linear operator $f\colon V \to V$. Then f is algebraic if and only if $f|U$ and $f|W$ are algebraic. Further, in this case $\mu_f = \mathrm{LCM}\,(\mu_{f|U}\,,\, \mu_{f|W})$. (**Remark:** See Exercise 10.8-(c) for an application. – **Hint:** Since $\mu_f(f \upharpoonright U) = \mu_f(f) \upharpoonright U = 0$ and $\mu_f(f \upharpoonright W) \upharpoonright = \mu_f(f) \upharpoonright W = 0$, clearly (by definition of minimal polynomial), $\mu_{f|U}$ and $\mu_{f|W}$ both divide μ_f. On the other hand put $\mu := \mathrm{LCM}\,(\mu_{f|U}\,,\, \mu_{f|W})$. Then $\mu(f) \upharpoonright U = \mu(f \upharpoonright U) = 0$ and $\mu(f) \upharpoonright W = \mu(f \upharpoonright W) = 0$, since μ is a multiple of both $\mu_{f|U}$ and $\mu_{f|W}$. Now, since $V = U + W$, it follows that $\mu(f) = 0$. Therefore (by definition of μ_f) μ_f divides μ.)

T10.15 Let $f\colon V \to V$ be an operator and let μ be the minimal polynomial of the restriction of f on $\mathrm{im}\, f$. Then either μ or $X\cdot\mu$ is the minimal polynomial of f. In particular, an operator f of finite rank r is algebraic and the degree of its minimal polynomial is $\leq r+1$. (**Hint :** Note that for the minimal polynomial μ_f of f, the operator $\mu_f(f) = 0$ and hence $\mu_f(f \upharpoonright \mathrm{Im}\, f) = \mu_f(f) \upharpoonright \mathrm{Im}\, f = 0$. Therefore $\mu = \mu_{f \upharpoonright \mathrm{Im}\, f}$ divides μ_f. On the other hand $(X\cdot\mu)(f) = f \circ \mu(f) = \mu(f) \circ f = 0$, since $\mu(f) \upharpoonright \mathrm{Im}\, f = 0$. This proves that μ_f divides $X\cdot\mu$ and hence the only possibilities are either $\mu_f = \mu$ or $\mu_f = X\cdot\mu$.)

T10.16 Let f be an invertible operator on the K-vector space V. Show that $\lambda \in K$ is an eigen-value (respectively a spectral-value) of f if and only if $1/\lambda$ is an eigen-value (respectively spectral-value) of f^{-1}, i. e. $\text{e-Spec}\,(f^{-1}) = (\text{e-Spec}\, f)^{-1} := \{\lambda^{-1} \mid \lambda \in \text{e-Spec}\, f\}$ and $\mathrm{Spec}\,(f^{-1}) = (\mathrm{Spec}\, f)^{-1} := \{\lambda^{-1} \mid \lambda \in \mathrm{Spec}\, f\}$.

T10.17 Let f and g be operators on the K-vector space V. Then show that

(a) The non-zero eigen-value s of fg and gf are same.

(b) The non-zero spectral-values of fg and gf are same. (**Hint :** For $a \in K^\times$, $fg - a\mathrm{id}$ is invertible if and only if $gf - a\mathrm{id}$ invertible. In this case $(gf - a\mathrm{id})^{-1} = a^{-1}\big(g\,(fg - a\mathrm{id})^{-1} f - \mathrm{id}\big)$.)

(c) Given an example such that the eigen-value s (resp. spectral-values) of fg and gf are not same. (**Hint :** Let $f, g : V := K[X] \to V = K[X]$ be the K-linear operators on the K-vector space $V = K[X]$ of polynomials over K (with basis X^n, $n \in \mathbb{N}$) defined by $f(X^n) := X^{n+1}$, $n \in \mathbb{N}$ and $g(X^n) := X^{n-1}$, for $n \geq 1$ and $g(X^0) = g(1) = 0$, i. e. $f := \lambda_X$ is the left multiplication by X and $g(P) := (P - P(0))/X$ for $PinK[X]$. Then 0 is an eigen-value (and hence a spectral-value) of fg, since $(fg)(1) = f(0) = 0 = 0\cdot 1$, but 0 is not an eigen-value (and moreover, not a sspectral-value) of gf, since $0\cdot\mathrm{id}_V - gf = gf = \mathrm{id}_V$ because $(gf)(X^n) = g(X^{n+1}) = X^n$ for all $n \in \mathbb{N}$.)

T10.18 Let $f\colon V \to V$ be a K-linear operator on the K-vector space V and let $U \subseteq V$ be an f-invariant subspace of V. Further, let $\overline{f} : V/U \to V/U$ be the operator on V/U induced by f. Then

(a) Show that every eigen-value of $f|U$ is an eigen-value of f and every eigen-value of f is an eigen-value of $f|U$ or of $\overline{f}$.

(b) The same statement as in the part (a) for the spectral-values, i. e.

$$\operatorname{Spec} f|U \subseteq \operatorname{Spec} f \subseteq \operatorname{Spec}\,(f|U) \cup \operatorname{Spec} \overline{f}\,.$$

(c) If f is algebraic, then $\operatorname{Spec} f = \operatorname{Spec}\,(f|U) \cup \operatorname{Spec} \bar{f}$.

T10.19 Let $f : V \to V$ be a K-linear operator and let V be the direct sum of the f-invariant subspaces V_i, $i \in I$. Show that

(a) The set of all eigen-values of f is the union of the set of all eigen-value s of $f|V_i$, $i \in I$, i.e.

$$\text{e-Spec}\,(f) = \bigcup_{i\in I} \text{e-Spec}\,(f|V_i)\,.$$

(b) For the spectral-values the analogous statement as in the part (a) holds, i.e.

$$\operatorname{Spec} f = \bigcup_{i\in I} \operatorname{Spec}\,(f|V_i)\,.$$

(c) Let λ_X denote the multiplication by the indeterminate X on the K-vectors space

(i) $V = K[X]$ of polynomials over K, then $\text{e-Spec}\,(\lambda_X) = \emptyset$ and $\operatorname{Spec}\,(\lambda_X) = K$.
(ii) $V = K(X)$ of rational functions over K, then $\text{e-Spec}\,(\lambda_X) = \operatorname{Spec}\,(\lambda_X) = \emptyset$.
(iii) $V = \{P/Q \in K(X) \mid P, Q \in K[X], Q(0) \neq 0\}$, then $\text{e-Spec}\,(\lambda_X) = \emptyset$ and $\operatorname{Spec}\,(\lambda_X) = \{0\}$.
(iv) $V = K[\![X]\!]$ of formal power series K, then $\text{e-Spec}\,(\lambda_X) = \emptyset$ and $\operatorname{Spec}\,(\lambda_X) = \{0\}$.

T10.20 Let $f : V \to V$ be an operator on the K-vector space V and let $P \in K[X]$ be a non-constant polynomial. Then show that

(a) If λ is an eigen-value (respectively spectral-value) of f, then $P(\lambda)$ is an eigen-value (respectively a spectral-value) of $P(f)$, i. e. $P(\text{e-Spec}\,(f)) \subseteq \text{e-Spec}\,P(f)$ and $P(\operatorname{Spec}\,(f)) \subseteq \operatorname{Spec} P(f)$. (**Hint :** Let $\lambda \in K$. Then λ is a zero of the polynomial $P(X) - P(\lambda) \in K[X]$ and hence $P(X) - P(\lambda) = (X - \lambda) \cdot Q(X)$ for some $Q \in K[X]$. Therefore $P(\lambda)\mathrm{id}_V - P(f) = (\lambda\,\mathrm{id}_V - f) \circ Q(f) = Q(f) \circ (\lambda\,\mathrm{id}_V - f)$ and hence if $(\lambda\,\mathrm{id}_V - f)$ is not injective (respectively not surjective), then $P(\lambda)\mathrm{id}_V - P(f)$ is not injective (respectively not surjective).)

(b) If K is *algebraically closed*,[6] then every eigen-value (respectively every spectral-value) of $P(f)$ of the form $P(\lambda)$ with an eigen-value (respectively a spectral-value) λ of f, i. e. $P(\text{e-Spec}\,(f)) = \text{e-Spec}\,P(f)$ and $P(\operatorname{Spec}\,(f)) \subseteq \operatorname{Spec} P(f)$. (**Hint :** Let $\mu \in K$ and let $P(X) - \mu = c(X - \lambda_1)\cdots(X - \lambda_n)$ with $c, \lambda_1, \ldots \lambda_n \in K$ (since K is algebraically closed. Therefore $\mu\mathrm{id}_V - P(f) = (-1)^{n-1}c(\lambda_1\mathrm{id}_V - f) \circ \cdots \circ (\lambda_n\mathrm{id}_V - f)$ and hence if $\lambda_i \notin \text{e-Spec} f$ (respectively, $\lambda_i \notin \operatorname{Spec} f$), then $\mu \notin \text{e-Spec}\,P(f)$ (respectively, $\mu \notin \operatorname{Spec} P(f)$).)

T10.21 Let f and g be operators on the K-vector space V with $[f, g] := fg - gf = a\,\mathrm{id}_V$ and let $a \neq 0$ in K. Show that if λ is an eigen-value of gf with the eigen-vector $x \in V$, then $gf(g^n(x)) = (\lambda + na)\,g^n(x)$, $n \in \mathbb{N}$. In particular, if $g^n(x) \neq 0$, then $\lambda + na$ is also an eigen-value of gf. Moreover, if g is invertible, then $\lambda + na$ is an eigen-value of gf with the eigen-vector $g^n(x)$ for $n \in \mathbb{Z}$. (**Hint :** By the way the relation $fg - gf = a\,\mathrm{id}_V$ with $a \neq 0$ is possible only in the case of a field characteristic 0 and only if V is either 0 or infinite dimensional. Otherwise, $(\operatorname{Dim} V) \cdot a = \operatorname{Tr}(a\,\mathrm{id}_V) = \operatorname{Tr}(fg) - \operatorname{Tr}(gf) = 0$ is a contradiction. It follows that there is no finite dimensional subspace $0 \neq U \subseteq V$ which is invariant under both f as well as g. In particular, f and g have no common eigen-vectors.)

T10.22 Let $f : V \to V$ be an operator on the K-vector space with the dual operator $f^* : V^* \to V^*$. Then show that

[6] A field K is called an a l g e b r a i c a l l y c l o s e d if every non-constant polynomial $P \in K[X]$ has a zero in K. For example, by the Fundamental Theorem of Algebra (see Footnote 2) the field $\mathbb{C}$ of complex numbers is algebraically closed. But the fields $\mathbb{Q}$, $\mathbb{R}$ and finite fields are not algebraically closed.

(a) A subspace U of V is f-invariant if and only if U° is f^*-invariant. (**Hint :** Suppose that $f(U) \subseteq U$ and $e \in U^\circ$. Then $e(x) = 0$ for all $x \in U$ and hence $(f^*(e))(x) = e(f(x)) = 0$ for all $x \in U$, since $f(x) \in U$ for all $x \in U$, i. e. $f^*(e) \in U^\circ$. This proves that $f^*(U^\circ) \subseteq U^\circ$. Conversely, suppose that $f^*(U^\circ) \subseteq U^\circ$ and let $x \in U$. For every $e \in U^\circ$, we have $f^*(e) \in U^\circ$ and hence $e(f(x)) = (f^*(e))(x) = 0$. Therefore every $e \in V^*$ which vanish on U also vanish on $f(x)$ and hence $f(x) \in U$ by Theorem 5.G.7. This proves that $f(U) \subseteq U$.)

(b) If a subspace W of V^* is f^*-invariant, then ${}^\circ W$ is f-invariant. If V is finite dimensional, then the converse hold. (**Hint :** Suppose that $f^*(W) \subseteq W$ and let $x \in {}^\circ W$. Then for every $e \in W$, we have $f^*(e) \in W$ and hence $e(f(x)) = (f^*(e))(x) = 0$, since $x \in {}^\circ W$. Therefore $f({}^\circ W) \subseteq {}^\circ W$. Conversely, suppose that V is finite dimensional and $f({}^\circ W) \subseteq {}^\circ W$. Then by Theorem 5.G.10 $({}^\circ W)^\circ = W$ and hence by the part (a) $f^*(W) = f^*(({}^\circ W)^\circ) \subseteq ({}^\circ W)^\circ = W$.)

(c) $\operatorname{Spec} f^* = \operatorname{Spec} f$ and in general e-Spec $f^* \neq$ e-Spec f (Example?).

T10.23 Let V be a n-dimensional vector space over a field K and let $\Delta \in \operatorname{Alt}_K(n,V)$ be an n-alternating linear form $V^n \to K$. For $f \in \operatorname{End}_K(V)$ and $x_1, \ldots, x_n \in V$, show that

$$\operatorname{Tr}(f) \cdot \Delta(x_1, \ldots, x_n) = \sum_{i=1}^{n} \Delta(x_1, \ldots, x_{i-1}, f(x_i), x_{i+1}, \ldots, x_n)\,.$$

T10.24 Let $f : V \to V$ be an operator on the finite dimensional K-vector space V and U be an f-invariant subspace of V. Then show that

$$\operatorname{Tr} f = \operatorname{Tr}(f \upharpoonright U) + \operatorname{Tr} \overline{f}\,,$$

where $\overline{f}$ is the operator $V/U \to V/U$ induced by f. In particular,

$$\operatorname{Tr} f = \operatorname{Tr}(f \upharpoonright \operatorname{Im} f) + \operatorname{Tr}(\overline{f}) \quad \text{with} \quad \overline{f}\colon V/\operatorname{Ker} f \longrightarrow V/\operatorname{Ker} f\,.$$

(**Hint:** By 11.A.8 we have $\chi_f = \chi_{f|U} \cdot \chi_{\overline{f}}$. – **Remark:** The last equation is used to define t r a c e o f a n o p e r a t o r o f f i n i t e r a n k on not necessary on finite dimensional vector spaces.)

T10.25 Let $f : V \to V$ be an operator on the finite dimensional K-vector space $V \neq 0$. Show that the following statements are equivalent:

(i) χ_f is a prime polynomial in $K[X]$.

(ii) 0 and V are the only f-invariant subspaces of V.

(iii) Every non-zero $x \in V$ is a cyclic vector (see Exercise 10.8-(d)) for f.

(**Hint:** If U is an f-invariant subspace of V with $0 < m := \operatorname{Dim}_K U < \operatorname{Dim}_K V$, then $\chi_f = \chi_{f|U} \cdot \chi_{\overline{f}}$ by 11.A.8 and $\deg \chi_{f|U} = \operatorname{Dim}_K U = m$ and hence $\chi_{f|U}$ is a proper divisor of χ_f, in particular, χ_f cannot be a prime polynomial. Conversely, if χ_f is not a prime polynomial and if P is a proper prime divisor of χ_f, then by 11.A.12 there exists an f-invariant subspace U of V of dimension $\operatorname{Dim}_K U = \deg P < \deg \chi_f = \operatorname{Dim}_K V$.)

T10.26 Let $f : V \to V$ be an operator on the finite dimensional K-vector space V. Show that

(a) If f is cyclic (see Exercise 10.8-(d)) with the characteristic polynomial $\chi := \chi_f$, then V has exactly

$$\prod_{\pi \in \mathbb{P}(K[X])} (\mathrm{v}_\pi(\chi) + 1)$$

f-invariant subspaces and restrictions of f to each one of these subspaces is again a cyclic operator, where $\mathbb{P}(K[X])$ denote the set of all monic prime polynomials in $K[X]$ and v_π denote the π-exponents.

(b) If K is infinite and if V has only finitely many f-invariant subspaces, then f is a cyclic operator. (**Hint :** Use Exercise 2.2.)

T10.27 Let $f : V \to V$ be a cyclic operator (see Exercise 10.8-(d)) on the finite dimensional K-vector space V of dimension n with the cyclic vector $x \in V$. Then the dual operator $f^* : V^* \to V^*$

is also a cyclic operator on the dual space V^* with a cyclic vector $(f^{n-1}(x))^*$, where $(f^{n-1}(x))^*$ belong to the dual basis of V^* with respect to the basis $x, f(x), \ldots, f^{n-1}(x)$ of V.

T10.28 Let $f: V \to V$ be an operator on the finite dimensional K-vector space V.

(a) Let v_i, $i \in I$ be a K-basis of V. Show that $\operatorname{Tr} f = \sum_{i\in I} v_i^*(f(v_i))$. (**Hint :** Let $\mathfrak{M}_{\mathfrak{v}}^{\mathfrak{v}}(f) = (a_{ij})_{(i,j)\in I\times I}$ be the matrix of f with respect to the basis $\mathfrak{v} = \{v_i \mid i \in I\}$, i. e. $f(v_j) = \sum_{i\in I} a_{ij} v_i$. Therefore $v_j^*(f(v_j)) = v_j^*(\sum_{i\in I} a_{ij} v_i) = \sum_{i\in I} a_{ij} v_j^*(v_i) = \sum_{i\in I} a_{ij}\delta_{ij} = a_{jj}$ and $\sum_{j\in I} v_j^*(f(v_j)) = \sum_{j\in I} a_{jj} = \operatorname{Tr}(f)$.)

(b) If Rank $f \leq 1$, then show that f is nilpotent if and only if $\operatorname{Tr} f = 0$. (**Hint :** By Test-Exercise T10.9 the characteristic polynomial $\chi_f = X^{n-1}(X - \operatorname{Tr}(f))$.)

T10.29 Let K be a field and let $n \in \mathbb{N}^*$. Then

(a) Show that the c o m m u t a t o r s $[\mathfrak{A}, \mathfrak{B}] := \mathfrak{A}\mathfrak{B} - \mathfrak{B}\mathfrak{A}$, $\mathfrak{A}, \mathfrak{B} \in \mathrm{M}_n(K)$, generate a subspace of codimension 1 in $\mathrm{M}_n(K)$. This subspace is the kernel of the trace function $\operatorname{Tr}: \mathrm{M}_n(K) \to K$.

(b) Show that every K-linear form $h: \mathrm{M}_n(K) \to K$ with $h(\mathfrak{A}\mathfrak{B}) = h(\mathfrak{B}\mathfrak{A})$ for all $\mathfrak{A}, \mathfrak{B} \in \mathrm{M}_n(K)$ is a scalar multiple of the trace function on $\mathrm{M}_n(K)$.

T10.30 Let $n \in \mathbb{N}$ and let K be a field with $k\,1_K \neq 0$ for $k = 1, \ldots, n$.

(a) For every operator $f: V \to V$ with $\operatorname{Tr} f = 0$ on a n-dimensional K-vector space V, show that there exists a basis $v_1, \ldots, v_n$ of V with $v_i^*(f(v_i)) = 0$, $i = 1, \ldots, n$. (**Hint:** By induction on k show that : there exist linearly independent vectors $v_1, \ldots, v_k$ and a subspace W_k of V such that

$$Kv_1 \oplus \cdots \oplus Kv_k \oplus W_k = V \quad \text{and} \quad f(v_i) \in \sum_{j\neq i} Kv_j + W_k.$$

Suppose that $k = 1$. If every element of V is an eigen-vector of f, then by Exercise 10.3 f is the homothecy $a\mathrm{id}_V$, $a \in K$ and it follows that $0 = \operatorname{Tr} f = n \cdot a$. Therefore $a = 0$ and $f = 0$, in this case the assertion is trivial. Otherwise, there exists a vector $v_1 \in V$ with $f(v_1) \notin Kv_1$. We extend $v_1, f(v_1)$ to a basis $v_1, f(v_1), w_1, \ldots, w_{n-2}$ of V and take W_1 the subspace of V generated by $f(v_1), w_1, \ldots, w_{n-2}$. With this the required assertion holds.

For the inductive step rom k to $k+1$, consider the map $p \circ f|W_k$, where p projection onto W_k along $\sum_{j=1}^{k} Kv_j$. Extend $v_1, \ldots, v_k$ to a basis $v_1, \ldots, v_k, w_1, \ldots, w_{n-k}$. Then removing the first k rows and first k columns from the matrix of f with respect to this basis, we obtain the matrix of $p \circ f|W_k$ with respect to the basis $w_1, \ldots, w_{n-k}$. Since the first k digonal elements of the matrix of f are 0 by construction and since $\operatorname{Tr} f = 0$, it follows that $\operatorname{Tr}(p \circ f|W_k) = 0$.

If every non-zero element of W_k is an eigen-vector of $p \circ f|W_k$, then by Exercise 10.3 $p \circ f|W_k$ is a homothecy $a \cdot \mathrm{id}_{W_k}$, $a \in K$ and it follows that $0 = \operatorname{Tr}(p \circ f|W_k) = (n-k) \cdot a$ and hence $a = 0$ by hypothesis on K. Therefore $p \circ f|W_k = 0$,i. e. $f(W_k) \subseteq Kv_1 \oplus \cdots \oplus Kv_k$. We can take arbitrary non-zero $v_{k+1} \in W_k$ and W_{k+1} a complement of Kv_{k+1} in W_k.

Otherwise there exists $v_{k+1} \in W_k$ such that $(p \circ f|W_k)(v_{k+1}) \notin Kv_{k+1}$ and so $f(v_{k+1}) \notin Kv_1 \oplus \cdots \oplus Kv_k \oplus Kv_{k+1}$. We extend $v_1, \ldots, v_k, v_{k+1}, f(v_{k+1})$ to a basis $v_1, \ldots, v_k, v_{k+1}, f(v_{k+1}), w_1, \ldots, w_{n-k-1}$ of V and take W_{k+1} the subspace of W_k generated by $f(v_{k+1}), w_1, \ldots, w_{n-k-1}$. With this the required assertion holds.

Now, in the case $k = n$, $W_n = 0$ and hence $v_1, \ldots, v_n$ is a basis of V such that $f(v_j) = \sum_{j\neq i} a_{ij} v_j$, i. e. the diagonal elements of the matrix of f with respect to this basis are all 0.)

(b) Show that every matrix $\mathfrak{A} \in \mathrm{M}_n(K)$ with $\operatorname{Tr}\mathfrak{A} = 0$ is a commutator, i.e. is of the form $[\mathfrak{B}, \mathfrak{C}] = \mathfrak{B}\mathfrak{C} - \mathfrak{C}\mathfrak{B}$. (**Hint:** By part (a) above the matrix $\mathfrak{A}$ is similar to the matrix $\mathfrak{A}'$ whose diagonal entries are all 0, i. e. there exists an invertible matrix $\mathfrak{D} \in \mathrm{M}_n(K)$ such that $\mathfrak{A} = \mathfrak{D}\mathfrak{A}'\mathfrak{D}^{-1}$. It is enough to show that there are matrices $\mathfrak{B}, \mathfrak{C} \in \mathrm{M}_n(K)$ such that $[\mathfrak{B}, \mathfrak{C}] = \mathfrak{A}'$. For, then $\mathfrak{A} = \mathfrak{D}\mathfrak{A}'\mathfrak{D}^{-1} = \mathfrak{D}(\mathfrak{B}\mathfrak{C} - \mathfrak{C}\mathfrak{B})\mathfrak{D}^{-1} = (\mathfrak{D}\mathfrak{B}\mathfrak{D}^{-1})(\mathfrak{D}\mathfrak{C}\mathfrak{D}^{-1} - (\mathfrak{D}\mathfrak{C}\mathfrak{D}^{-1})(\mathfrak{D}\mathfrak{B}\mathfrak{D}^{-1} = [\mathfrak{D}\mathfrak{B}\mathfrak{D}^{-1}\ ,\ \mathfrak{D}\mathfrak{C}\mathfrak{D}^{-1}]$. Therefore, without loss of generality assume that all main-diagonal entries of $\mathfrak{A} = (a_{ij})$ are 0. Since $\#K > n$ by hypothesis on K, there exists distinct elements $b_1, \ldots, b_n \in K$. Then for the diagonal matrix $\mathfrak{B} = \operatorname{Diag}(b_1, \ldots, b_n)$, and an arbitrary matrix $\mathfrak{C} = (c_{ij}) \in \mathrm{M}_n(K)$, we have

$$\begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} - \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b_1c_{11} & b_1c_{12} & \cdots & b_1c_{1n} \\ b_2c_{21} & b_2c_{22} & \cdots & b_2c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_nc_{n1} & b_nc_{n2} & \cdots & b_nc_{nn} \end{pmatrix} - \begin{pmatrix} b_1c_{11} & b_2c_{12} & \cdots & b_nc_{1n} \\ b_1c_{21} & b_2c_{22} & \cdots & b_nc_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_1c_{n1} & b_2c_{n2} & \cdots & b_nc_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (b_1-b_2)c_{12} & \cdots & (b_1-b_n)c_{1n} \\ (b_2-b_1)c_{21} & 0 & \cdots & (b_2-b_n)c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (b_n-b_1)c_{n1} & b_nc_{n2} & \cdots & 0 \end{pmatrix}.$$

Now, one can take $c_{ij} := a_{ij}/(b_i - b_j)$ for $i \neq j$ and $c_{ii} = 0$, so that the equation $[\mathfrak{B}, \mathfrak{C}] = \mathfrak{A}$ holds.)

T10.31 Let V be a finite dimensional K-vector space.

(a) For a projection p of V, show that $\operatorname{Tr} p = \operatorname{Rank} p \,(= (\operatorname{Rank} p) 1_K)$. (**Hint :** Use Test-Exercise T8.9-(a).)

(b) Suppose that $m \cdot 1_K \neq 0$ for $1 \leq m \leq \operatorname{Dim}_K V$. Further, let $p_1, \ldots, p_r$ be projections of V with $p_1 + \cdots + p_r = \operatorname{id}_V$. Further, suppose that either $\operatorname{Char} K = 0$ or $\sum_{i=1}^{r} \operatorname{Rank} p_i - \operatorname{Dim}_K V < \operatorname{Char} K$, if $\operatorname{Char} K > p$. Then show that $p_i p_j = \delta_{ij} p_i$ for $1 \leq i, j \leq r$ and in particular, V is the direct sum of the subspaces $\operatorname{Im}\, p_i$, $i = 1, \ldots, r$. (**Hint:** Since $p_1 + cdots + p_r = \operatorname{id}_V$, we have $\operatorname{Im} p_1 + \cdots + \operatorname{Im} p_r = V$ and hence $\operatorname{Dim}_K V = \operatorname{Tr}(\operatorname{id}_V) = \operatorname{Tr}(p_1) + \cdots + \operatorname{Tr} p_r == \operatorname{Rank} p_1 + \cdots + \operatorname{Rank} p_r$. Therefore by the assumption on the characteristic of K, the equality $\operatorname{Dim}_K V = \operatorname{Rank} p_1 + \cdots + \operatorname{Rank} p_r$ also hold in $\mathbb{N}$ and hence the sum $V = \operatorname{Im}_1 \oplus \cdots \oplus \operatorname{Im} p_r$ is direct. Therefore $\operatorname{Im} p_j \subseteq \operatorname{Ker} p_i$ for all $i \neq j$ and hence $p_i \circ p_j = 0$ for all i, j, $i \neq j$. Further, $p_i \circ p_i = p_i$, since p_i is a projection, for all $i = 1, \ldots, r$.)

(c) Suppose that a finite group G operates on V as the group of K- automorphisms and that $|G| \cdot 1_K \neq 0$ in K. Then show that

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma$$

is a projection of V onto $\operatorname{Fix}_G V$ (see also Example 6.E.10) and the equality (in K)

$$\operatorname{Dim}_K \operatorname{Fix}_G V = \frac{1}{|G|} \sum_{\sigma \in G} \operatorname{Tr} \sigma .$$

(**Hint:** For a fixed $\tau \in G$, note that $G = \{\tau\sigma \mid \sigma \in G\}$. Therefore for $p := \frac{1}{\#G} \sum_{\sigma \in G} \sigma$, we have

$$p^2 = \frac{1}{(\#G)^2} \sum_{\sigma \in G} \sigma \sum_{\tau \in G} \tau\sigma = \frac{\#G}{(\#,G)^2} \sum_{\sigma \in G} \sigma = \frac{1}{\#G} \sum_{\sigma \in G} \sigma = p .$$

Therefore p is a projection of V. For a $x \in \operatorname{Fix}_G V$, $\sigma(x) = x$ for all $\sigma \in G$ and hence $p(x) = \dfrac{1}{\#G} \displaystyle\sum_{\sigma \in G} x = x$. Conversely, for $y = p(x) \in \operatorname{Im} p$, it is immediate that $\tau(y) = \dfrac{1}{\#G} \displaystyle\sum_{\sigma \in G} \tau\sigma(x) = \dfrac{1}{\#G} \displaystyle\sum_{\sigma \in G} \sigma(x) = p(x) = y$ for all $\tau \in G$. Therefore $\operatorname{Dim}_K \operatorname{Fix}_G V = \operatorname{Dim}_K \operatorname{Im} p = \operatorname{Rank} p = \operatorname{Tr} p = \dfrac{1}{\#G} \displaystyle\sum_{\sigma \in G} \operatorname{Tr} \sigma$.)

T10.32 (Jacobson-Lemma) Let K be a field and let f, g be operators on the n-dimensional K-vector space V with $\big[f, [f, g]\big] = 0$. Suppose that $m \cdot 1_K \neq 0$ for $1 \leq m \leq \operatorname{Dim}_K V$. Then $[f, g]$ nilpotent. (**Hint :** The condition $\big[f, [f, g]\big] = 0$ is equivalent with $f[f, g] = [f, g]f$ and hence f commute with the powers $[f, g]^n$, $n \in \mathbb{N}$. It follows that $[f, g]^n = (fg - gf)[f, g]^{n-1} = fg[f, g]^{n-1} - gf[f, g]^{n-1} = fg[f, g]^{n-1} - g[f, g]^{n-1} f = \big[f, g[f, g]^{n-1}\big]$. Now, since $[f, g]^{n-1}$ are also commutators, they have trace 0 and hence $[f, g]$ is nilpotent by Exercise 10.5-(a).)

T10.33 Let $\mathfrak{A}$ be a $n \times n$-matrix over the field K. Suppose that the sum of elements of every row of $\mathfrak{A}$ is equal to $\lambda \in K$. Then show that λ is an eigen-value of $\mathfrak{A}$ with the eigen-vector ${}^t(1, 1, \ldots, 1) \in K^n$. If all the column-sum of $\mathfrak{A}$ are equal to λ, then λ is an eigen-value of $\mathfrak{A}$.

(**Hint:** Clearly, $\mathfrak{A}\,{}^{\mathrm{tr}}(1,\ldots,1) = {}^{\mathrm{tr}}(\lambda,\ldots,\lambda) = \lambda\,{}^{\mathrm{tr}}(1,\ldots,1)$, i. e. λ is an eigen-value of $\mathfrak{A}$. – **Remark:** An eigen-vector corresponding to this eigen-value is, in general, no so easy to give explicitly.)

T10.34 Let $\mathfrak{A} \in \mathrm{M}_{m,n}(K)$ and $\mathfrak{B} \in \mathrm{M}_{n,m}(K)$, $m \geq n$. Show that $\chi_{\mathfrak{A}\mathfrak{B}} = X^{m-n}\chi_{\mathfrak{B}\mathfrak{A}}$. (**Hint:** Fill the matrices $\mathfrak{A}$ and $\mathfrak{B}$ with zeroes to get square $m \times m$-matrices. $(\mathfrak{A}\ 0)\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix} = \mathfrak{A}\mathfrak{B}$ and $\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix}(\mathfrak{A}\ 0) = \begin{pmatrix}\mathfrak{B}\mathfrak{A} & 0\\0 & 0\end{pmatrix}$. Therefore the characteristic polynomial $\chi_{\mathfrak{A}\mathfrak{B}}$ is equal to that of $(\mathfrak{A}\ 0)\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix}$ and hence the characteristic polynomial of $\begin{pmatrix}\mathfrak{B}\\0\end{pmatrix}(\mathfrak{A}\,0)$ is equal to $\mathrm{Det}\begin{pmatrix}X\mathfrak{E}_n - \mathfrak{B}\mathfrak{A} & 0\\0 & X\mathfrak{E}_{m-n}\end{pmatrix} = X^{m-n}\mathrm{Det}\,(X\mathfrak{E}_n - \mathfrak{B}\mathfrak{A}) = X^{m-n}\chi_{\mathfrak{b}\mathfrak{A}}$ by Exercise 10.7-(b).)

T10.35 **(a)** Let V be a finite dimensional vector space over a field K and let $f \in \mathrm{End}_K V$. Further, let $\mathrm{L}_f : \mathrm{End}_K V \to \mathrm{End}_K V$, $g \mapsto fg$ (respectively $\mathrm{R}_f : \mathrm{End}_K V \to \mathrm{End}_K V$, $g \mapsto gf$ be the left-translation by f. Show that

$$\chi_{\mathrm{L}(f)} = \chi_{\mathrm{R}(f)} = \big(\chi_f\big)^n\,,\ \mathrm{Tr}\,\mathrm{L}(f) = \mathrm{Tr}\,\mathrm{R}(f) = n\cdot\mathrm{Tr}\,f \text{ and } \mathrm{Det}\,\mathrm{L}(f) = \mathrm{Det}\,\mathrm{R}(f) = (\mathrm{Det}\,f)^n\,.$$

(See also Example 11.A.27).

(b) Show that the characteristic polynomial of a complex number z as an element of the $\mathbb{R}$-algebra $\mathbb{C}$ is $\chi_z = (X-z)(X-\bar{z})$. In particular, $\mathrm{N}_{\mathbb{R}}^{\mathbb{C}} z = z\bar{z} = |z|^2$ and $\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}} z = z + \bar{z} = 2\,\mathrm{Re}\,z$.

T10.36 Let f be an operator on a finite dimensional K-vector space and let $P \in K[X]$ be a polynomial. Show that $P(f)$ is invertible if and only if P and μ_f (or also P and χ_f) are relatively prime. (**Hint :** Let $Q := \gcd(P,\mu_f)$. If $Q = 1$, then $SP + T\mu_f = 1$ for some polynomials $S,T \in K[X]$ and hence $\mathrm{id} = S(f)P(f) + T(f)\mu_f(f) = S(f)P(f)$, i. e. $P(f)$ is invertible with inverse $S(f)$. Conversely, if $Q \neq 1$, then $\mu_f = R\cdot Q$, $P = P'\cdot Q$ with $R, P' \in K[X]$ and $\deg R < \deg \mu_f$ and hence $R(f) \neq 0$ and $Q(f) \neq 0$, but $0 = \mu_f(f) = R(f)\circ Q(f) = Q(f)\circ R(f)$. Therefore $Q(f)$ is not injective and hence $P(f) = P'(f)\circ Q(f)$ is also not injective. In particular, $P(f)$ is not invertible.)

T10.37 Let K be a field.

(a) Let P and Q be monic polynomials over the field K. Suppose that $\deg P = n$, Q is a divisor of P and moreover that P and Q have the same prime factors in $K[X]$. Then show that on every n-dimensional K-vector space V there exists an operator $f \in \mathrm{End}_K V$ with characteristic polynomial $\chi_f = P$ and minimal polynomial $\mu_f = Q$.

(b) Let S and S' be subsets of K with $S \subseteq S'$. Show that there exists a K-linear operator $f : V \to V$ on a K-vector space V such that e-Spec $f = S$ and Spec $f = S'$. (**Hint :** For each $a \in K$, let $g_a = -\lambda_a$ and $h_a := \lambda_{X-a}$ be operators on the K-vector space $K[\![X]\!]$. Then e-Spec $g_a = \{a\} =$ Spec g_a, e-Spec $h_a = \emptyset$ and Spec $h_a = \{a\}$, see Test-Exercise T10.19-(c). Let $g := (\oplus_{a\in S} g_a) : K^{(S)} \to K^{(S)}$ and $h := (\oplus_{a\in S'\setminus S} h_a) : K^{(S'\setminus S)} \to K^{(S'\setminus S)}$ be the direct sum of operators g_a, $a \in S$ and h_a, $a \in S'\setminus S$ respectively. Now it is easy to check that the operator $f := g \oplus h$ have the required properties. See Test-Exercise T10.19 also.)

T10.38 Show that an operator f on a $\mathbb{R}$-vector space has exactly one real eigen-value if and only if f^2 has an eigen-value ≥ 0. (**Hint :** $f^2 - a^2\mathrm{id} = (f - a\,\mathrm{id})(f + a\,\mathrm{id})$.)

T10.39 Let f be a $\mathbb{C}$-linear operator on the finite dimensional $\mathbb{C}$-vector space V, which we consider as $\mathbb{R}$-vector space. Then show that f is also $\mathbb{R}$-linear and

$$\chi_{f,\mathbb{R}} = \chi_{f,\mathbb{C}}\cdot\overline{\chi}_{f,\mathbb{C}}\,. \quad \text{(for a polynomial } P = \textstyle\sum a_i X^i \in \mathbb{C}[X]\text{, we put } \overline{P} := \sum \overline{a}_i X^i\text{)}$$

Further, for the minimal polynomials show that $\mu_{f,\mathbb{R}} = \mathrm{LCM}\big(\mu_{f,\mathbb{C}}\,,\,\overline{\mu}_{f,\mathbb{C}}\big)$.

T10.40 Let $\mathfrak{A} = (a_{ij}) \in \mathrm{M}_n(K)$ be a $n \times n$-matrix over the field K. Then

(a) Let $X_1,\ldots,X_n$ be indeterminates over K. For $1 \leq i_1 < \cdots < i_r \leq n$, show that the coefficient of

$X_{i_1}\cdots X_{i_r}$ in the polynomial

$$\begin{vmatrix} a_{11}+X_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}+X_n \end{vmatrix} \in K[X_1,\ldots,X_n]$$

is equal to the diagonal minor of $\mathfrak{A}$ obtained by removing the rows and columns numbered by $i_1,\ldots,i_r$. (**Hint :** Expand the determinant successively using the rows $i_1,\ldots,i_r$.)

(b) For $r=1,\ldots,n$, show that the coefficient a_r of X^r in the characteristic polynomial $\chi_{\mathfrak{A}}$ of $\mathfrak{A}$ is $(-1)^{n-r}$-times the sum of the diagonal minors of the order $n-r$ of $\mathfrak{A}$.

T10.41 Let $K\subseteq L$ be a field extension and let $\mathfrak{A}\in \mathrm{M}_n(K)\subseteq \mathrm{M}_n(L)$ be a matrix with an eigen-value $\lambda\in L-K$. Then there exists an eigen-vector $\mathfrak{x}\neq 0$ in L^n of $\mathfrak{A}$, i. e. $\mathfrak{A}\mathfrak{x}=\lambda\mathfrak{x}$; but there is no eigen-vector in K^n, i. e. $K^n\cap \mathrm{Ker}\,(\lambda\mathfrak{E}_n-\mathfrak{A})=0$.

T10.42 (Jacobi's Matrix) For $k=0,\ldots,n$, let

$$\mathfrak{D}_k := \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{k-1} & b_{k-1} \\ 0 & 0 & 0 & \cdots & c_{k-1} & a_k \end{pmatrix} \in \mathrm{M}_k(K)$$

and let $\mathrm{D}_k := \mathrm{Det}\,(\mathfrak{D}_k)$ (see exercise (13.30)). Put $\chi_k := \chi_{\mathfrak{D}_k}$. Show that

(a) $\chi_0=1$, $\chi_1=X-a_1$, $\chi_k=(X-a_k)\chi_{k-1}-b_{k-1}c_{k-1}\chi_{k-2}$ for all $k=2,\ldots,n$.

(b) If $K=\mathbb{R}$ and $b_kc_k>0$ for all $k=1,\ldots,n$, then χ_n has n-distinct real roots and the number of positive roots of χ_n is the number of changes in the sign of the sequence $1,-\mathrm{D}_1,\ldots,(-1)^n\mathrm{D}_n$.

T10.43 Let $\mathfrak{A}=(a_{ij})_{1\le i,j\le n}\in \mathrm{M}_n(K)$. Show that

$$\chi_{\mathfrak{A}} = X^n - s_1X^{n-1}+s_2X^{n-2}-\cdots+(-1)^n s_n,$$

where s_k is the sum of $\binom{n}{k}$ minors $\mathrm{Det}\,(\mathfrak{A}(i_1,i_2,\ldots,i_k))$, $1\le i_1<i_2<\cdots<i_k\le n$.

T10.44 Let $a,b,c\in\mathbb{C}$ with $bc\neq 0$ and let

$$\mathfrak{T}_n := \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{pmatrix} \in \mathrm{M}_k(K)\,,\ \text{for } k=0,\ldots,n.$$

Show that

(a) $\lambda_k = a+2\sqrt{bc}\,\cos\left(\dfrac{\pi k}{n+1}\right)$, $k=1,\ldots,n$ are eigen-values of $\mathfrak{T}_n$.

(b) For $k=1,\ldots,n$, the vector with i-th components $\left(\sqrt{\dfrac{c}{b}}\right)^{i-1}\sin\left(\dfrac{\pi k}{n+1}\right)$ $i=1,\ldots,n$, is an eigen-vector corresponding to the eigen-value λ_k. (**Hint :** We may assume that $a=0$. Let $\mu\in\mathbb{C}$ with $\mu^2\neq bc$ and let $T_n(\mu):=\mathrm{Det}\,(\mu\mathfrak{E}_n-\mathfrak{T}_n)$. Then show that $T_0(\mu)=1, T_1(\mu)=\mu$ and $T_{k+2}(\mu)=\mu T_{k+1}(\mu)-bcT_k(\mu)$ for all $k\ge 0$. Therefore by Test-Exercise T10.42 $T_n(\mu)=\dfrac{(\mu_1^{n+1}-\mu_2^{n+1})}{(\mu_1-\mu_2)}$ where μ_1 and μ_2 are distinct roots of the quadratic $X^2-\mu X+bc$. Now, determine μ so that $\mu_1^{n+1}=\mu_2^{n+1}$.)

T10.45 Let V be a n-dimensional vector space over a field K and let $f \in \mathrm{End}_K(V)$.

(a) If $\mathrm{Char}(K) = p > 0$ then, show that $\chi_{f^p}(X^p) = \big(\chi_f\big)^p$. In particular, $\mathrm{Tr}(f^p) = (\mathrm{Tr}(f))^p$. (**Hint :** For $\mathfrak{A} \in \mathrm{M}_n(A)$ we have $(X\mathfrak{E}_n - \mathfrak{A})^p = X^p\mathfrak{E}_n - \mathfrak{A}^p$. — This is a special case of the following more general exercise in part (b) below.)

(b) For $r \in \mathbb{N}^+$, prove that

$$\chi_{f^r}(X^r) = (-1)^{n(r-1)} \prod_{i=1}^{r} \chi_f(\zeta_i X)\,,$$

where ζ_i, $i = 1, \ldots r$ are the r-th roots of unity, i.e. $X^r - 1 = \prod_{i=1}^{r}(X - \zeta_i)$. Deduce that $\chi_{f^2}(X^2) = (-1)^n \chi_f(X)\chi_f(-X)$.

T10.46 Let $\mathfrak{A} \in \mathrm{M}_n(K)$ and let $\chi_{\mathfrak{A}} = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$. Show that

(a) $\mathrm{Adj}(\mathfrak{A}) = (-1)^{n+1}(\mathfrak{A}^{n-1} + a_{n-1}\mathfrak{A}^{n-2} + \cdots + a_1\mathfrak{E}_n)$.

(b) $\chi_{\mathrm{Adj}(\mathfrak{A})} = X^n + (-1)^n \sum_{i=1}^{n} a_i \left(\mathrm{Det}\,(\mathfrak{A})\right)^{i-1} X^{n-i}$, where $a_n := 1$.

T10.47 Let I be a finite indexed set. Let $R := K[X_{ij} \mid i,j \in I]$ (respectively, $Q := K(X_{ij} \mid i,j \in I\})$ be a polynomial algebra (respectively the field of rational functions) over a field K and let $\mathfrak{A} = \big(X_{ij}\big) \in \mathrm{M}_I(Q)$. Then the characteristic polynomial $\chi_{\mathfrak{A}} \in R[X]$ is a prime polynomial in $R[X]$.

T10.48 Let f, g be operators on a finite dimensional K-vector space V such that $\chi_f = \chi_g$. Then show that $\chi_{P(f)} = \chi_{P(g)}$ for every polynomial $P \in K[X]$. (**Hint :** It is enough to show that: if $\mathfrak{A} \in \mathrm{M}_n(K)$ and if $\mathfrak{B}$ is the companion matric of the polynomial $\chi_{\mathfrak{A}}$, then $\chi_{P(\mathfrak{A})} = \chi_{P(\mathfrak{B})}$ for all $P \in K[X]$. For this we may take $R := K[X_{ij}, Y_k \mid i,j \in I, k = 0,\ldots,m]$ (respectively, $Q := K(X_{ij}, Y_k \mid i,j \in I, k = 0,\ldots,m\})$ the polynomial algebra (respectively the field of rational functions) over K, $\mathfrak{A} := (X_{ij}) \in \mathrm{M}_I(Q)$ and $P = Y_0 + Y_1X + \cdots + Y_mX^m$. Now $\mathfrak{A}$ is similar to the companion matrix of $\mathfrak{A}$ by Test-Exercises T10.?? and T10.??.)

T10.49 Let $\mathfrak{A} \in \mathrm{M}_n(K)$. Show that the following equality holds in the field of rational functions $K(X)$ over K:

$$\mathrm{Tr}\,((X\mathfrak{E}_n - \mathfrak{A})) = \frac{\chi'_{\mathfrak{A}}}{\chi_{\mathfrak{A}}}\,, \quad \text{where} \quad \chi'_{\mathfrak{A}} = \frac{d}{dX}\chi_{\mathfrak{A}}\,.$$

******T10.50** (Shift operator on the space of sequences) On the K-vector space $K^{\mathbb{N}}$ of the sequences with values in a field K, let

$$\mathrm{s} : K^{\mathbb{N}} \to K^{\mathbb{N}} \quad F \mapsto \mathrm{s}(F) := F^- : n \mapsto F(n+1)\,,$$

denote the (left) shift operator.

(a) (Sequences with linear recursion equations) For a polynomial $P = a_0 + a_1X + \cdots + a_nX^m \in K[X]$, show that the kernel of the linear operator $P(\mathrm{s}) : K^{\mathbb{N}} \to K^{\mathbb{N}}$, $F \mapsto P(\mathrm{s})(F)$ is:

$$\mathrm{Ker}\, P(\mathrm{s}) = \{F \in K^{\mathbb{N}} \mid a_0F(n) + a_1F(n+1) + \cdots + a_nF(n+m) = 0 \ \text{ for all } \ n \in \mathbb{N}\}\,.$$

We shall say that the sequences in $\mathrm{Ker}\, P(\mathrm{s})$ satisfy the (linear) recursion equation corresponding to the (recursion) polynomial P.

1) Let $P \in K[X]$ be a monic polynomial of degree m. The operator $P(\mathrm{s}) : K^{\mathbb{N}} \to K^{\mathbb{N}}$ is surjective and the map $\mathrm{Ker}\, P(\mathrm{s}) \to K^m$, $F \mapsto (F(n))_{0 \le n < m}$ is an isomorphism of K-vector spaces.
In particular, $\mathrm{Dim}_K \mathrm{Ker}\, P(\mathrm{s}) = m = \deg P$. Moreover, the sequences $F_0, F_1, \ldots, F_{m-1} \in \mathrm{Ker}\, P(\mathrm{s})$ is a K-basis of $\mathrm{Ker}\, P(\mathrm{s})$ if and only if the Casorati's determinant

$$\mathrm{C}(F_0, F_1, \ldots, F_{m-1}) := \begin{vmatrix} F_0(0) & F_1(0) & \cdots & F_{m-1}(0) \\ F_0(1) & F_1(1) & \cdots & F_{m-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(m-1) & F_1(m-1) & \cdots & F_{m-1}(m-1) \end{vmatrix}$$

of $F_0, F_1, \ldots, F_{m-1}$ is non-zero.

2) Let $P = a_0 + a_1X + \cdots + a_{m-1}X^{m-1} + X^m \in K[X]$ be a monic polynomial of degree m. A sequence $F \in \operatorname{Ker} P(\mathrm{s})$ is a cyclic element (see Exercise 10.8-(d)) for the restriction operator $\mathrm{s}_{|\operatorname{Ker} P(\mathrm{s})} : \operatorname{Ker} P(\mathrm{s}) \to \operatorname{Ker} P(\mathrm{s})$ if and only if the *Hankel's determinat*

$$\mathrm{H}_m^{(0)}(F) := \begin{vmatrix} F(0) & F(1) & \cdots & F(m-1) \\ F(1) & F(2) & \cdots & F(m) \\ \vdots & \vdots & \ddots & \vdots \\ F(m-1) & F(m) & \cdots & F(2m-2) \end{vmatrix}$$

of F is non-zero.

(b) (*Geometric sequences*) Let $a \in K$ and let $P := (X-a)^m \in K[X]$, $m \in \mathbb{N}^+$. Consider the operator $\Delta_a := \mathrm{s} - a \cdot \mathrm{id} : K^{\mathbb{N}} \to K^{\mathbb{N}}$ on $K^{\mathbb{N}}$. Then $P(\mathrm{s}) = \Delta_a^m$.

1) Show that the sequences

$$\left(\binom{n}{i} a^{n-i}\right)_{n\in\mathbb{N}}, \quad i = 0, \ldots, m-1, \quad \left(\text{we put } \binom{n}{i} a^{n-i} = 0 \text{ for } n < i\right)$$

form a basis of the kernel $\operatorname{Ker} \Delta_a^m$. (**Hint:** The elements $(x-a)^i$, $i = 0, \ldots, m-1$, form a K-basis of the K-vector (quotient) space $K[x] := K[X]/K[X]P$. – **Remark:** If $a = 1$, then $\operatorname{Ker} \Delta_a^m$ is the arithmetic sequences of degree $< m$. For arbitrary a, these sequences are called the *geometric sequences of degree $< m$ with the quotients a.*)

2) If $Q := \sum_{i=0}^{m-1} d_i X^i \in K[X]$ is a polynomial of degree $< m$, then show that $F = (Q(n)a^n)_{n\in\mathbb{N}}$ is a geometric sequence of degree $< m$ with the quotients a and $\Delta_a^{m-1} F = (m-1)!\, a^{m-1} d_{m-1} F_0$, where F_0 is the standard geometric sequence $(a^n)_{n\in\mathbb{N}}$ with the quotients a.

3) If $a \neq 0$ and $\mathbb{Q} \subseteq K$ (equivalently, $\operatorname{Char} K = 0$), then the sequences

$$(n^i a^n)_{n\in\mathbb{N}}, \quad i = 1, \ldots, m-1,$$

form a basis of the K-vector space of the geometric sequences of degree $< m$ with the quotients a.

4) If $F \in K^{\mathbb{N}}$ is a geometric sequence of degree $< m$ with the quotients a, then the Hankel's determinant $\mathrm{H}_m^{(0)}(F) = (-1)^{\binom{m}{2}} \left(\Delta_a^{m-1} F(0)\right)^m$. In particular, if $Q = \sum_{i=0}^{m-1} d_i X^i \in K[X]$, then

$$\begin{vmatrix} Q(0) & Q(1) & \cdots & Q(m-1) \\ Q(1) & Q(2) & \cdots & Q(m) \\ \vdots & \vdots & \ddots & \vdots \\ Q(m-1) & Q(m) & \cdots & Q(2m-2) \end{vmatrix} = (-1)^{\binom{m}{2}} \left((m-1)!\right)^m d_{m-1}^m .$$

(c) Let $F \in K^{\mathbb{N}}$ and let $0 \neq P \in K[X]$. If F satisfy a recursion equation $P(\mathrm{s})F = 0$, then the minimal polynomial μ_F of F is also called the *minimal recursion polynomial of F.*

1) If $P(\mathrm{s})F = 0$ and $P \in K[X]$ is a monic polynomial of degree m, then $P = \mu_F$ if and only if the Hankel's determinnat $\mathrm{H}_m^{(0)}(F) \neq 0$. Moreover, in this case $\mathrm{s}^i F$, $i = 0, \ldots, m-1$, is a K-basis of the space of all sequences $G \in K^{\mathbb{N}}$ which satisfy the recursion equation $P(\mathrm{s})G = 0$.

2) For $a \in K$, the $(X-a)$-primary component $\mathrm{V}(a;\mathrm{s}) := \cup_{r\in\mathbb{N}} \mathrm{V}_f^r(a)$ of s, (where $\mathrm{V}_f^r(a) := \operatorname{Ker}(f - a \cdot \mathrm{id}_V)^r$), is the space of geometric sequences with the quotients a, see the part (b). The space $V^r(a,\mathrm{s})$, $r \in \mathbb{N}$, is the space of geometric sequences of degree $< r$ with the quotients a. If $\operatorname{Char} K = 0$ and $a \neq 0$, then it has the basis

$$\left(\binom{n}{i} a^{n-i}\right)_{n\in\mathbb{N}}, \quad i = 0, \ldots, r-1.$$

Moreover, $\left(n^i a^n\right)_{n\in\mathbb{N}}$, $i=0,\dots,r-1$ is also a basis. If $P=(X-a_1)^{\nu_1}\cdots(X-a_r)^{\nu_r}$ with pairwise distinct $a_1,\dots,a_r\in K$, then the space of sequences F which satisfy the recursion equation $P(\mathrm{s})(F)=0$ is the direct sum $\mathrm{V}^{\nu_1}(a_1;\mathrm{s})\oplus\cdots\oplus\mathrm{V}^{\nu_r}(a_r;\mathrm{s})$.

3) Let $P:=(X-a_1)\cdots(X-a_r)$ with pairwise distinct $a_1,\dots,a_r\in K$ and

$$\prod_{i=1,i\neq j}\frac{X-a_i}{a_j-a_i}=\frac{a}{(X-a_j)}\cdot\frac{1}{P'(a_j)}=\sum_{i=0}^{r-1}a_{ij}X^i\,,\quad j=1,\dots,r,\quad a_{ij}\in K\,.$$

If $F\in K^{\mathbb{N}}$ satisfy the recursion equation $P(\mathrm{s})F=0$ and the initial conditions $F(n)=c_n$, $n=0,\dots,r-1$, then the matrix-equation

$$F(n)=(c_0,\dots,c_{r-1})\cdot\begin{pmatrix} a_{01} & a_{11} & \cdots a_{0,r-1} & \\ a_{11} & a_{12} & \cdots a_{1,r-1} & \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,1} & a_{r-1,1} & \cdots a_{r-1,r-1} & \end{pmatrix}\cdot\begin{pmatrix} a_1^n\\ \vdots \\ a_r^n\end{pmatrix}$$

4) If $P\in K[X]$ be a monic quadratic polynomial with distinct zeros a_1,a_2, then for a sequence $F\in K^{\mathbb{N}}$ with $P(\mathrm{s})F=0$ and $F(0)=c_0$, $F(1)=c_1$, we have

$$F(n)=\frac{1}{a_2-a_1}\left((c_0a_2-c_1)a_1^n+(-c_0a_1+c_1)a_2^n\right)\,,\ n\in\mathbb{N}\,.$$

If P has a double zero a, then

$$F(n)=c_0a^n+(-c_1-c_0a)na^{n-1}\,,\ n\in\mathbb{N}\,.$$

For the Fibonacci sequence $(f_n)\in\mathbb{R}^{\mathbb{N}}$ with $f_0=f_1=1$ and $f_{n+2}=f_{n+1}+f_n$, $n\in\mathbb{N}$, we have the Binet's formula:

$$f_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\,,\ n\in\mathbb{N}\,.$$

(d) For a sequence $F\in K^{\mathbb{N}}$, show that the following statements are equivalent:

(i) The sequence F satisfy a linear recursion equation $P(\mathrm{s})F=0$, where $P\in K[X]$, $P\neq 0$.

(ii) There exists a $r\in\mathbb{N}$ such that all Hankel's determinants $\mathrm{H}_{r+1}^{(k)}(F)=0$ for all $k\in\mathbb{N}$, where for $k,m\in\mathbb{N}$, we put

$$\begin{aligned}\mathrm{H}_m^{(k)}(F)&:=\mathrm{H}_m^{(0)}(\mathrm{s}^kF)=\mathrm{C}(\mathrm{s}^kF,\mathrm{s}^{k+1}F,\dots,\mathrm{s}^{k+m-1}F)\\ &=\begin{vmatrix} F(k) & F(k+1) & \cdots & F(k+m-1)\\ F(k+1) & F(k+2) & \cdots & F(k+m)\\ \vdots & \vdots & \ddots & \vdots\\ F(k+m-1) & F(k+m) & \cdots & F(k+2m-2)\end{vmatrix}.\end{aligned}$$

– Suppose that these conditions are satisfied, then: if $m\in\mathbb{N}$ is the smallest natural number with $\mathrm{H}_{m+1}^{(k)}(F)=0$ for all $k\in\mathbb{N}$, then m is the degree of the minimal recursion polynomial P_F of F and $\mathrm{H}_m^{(0)}(F)\neq 0$. (**Hint :** For a proof of the implication (ii)⇒(i) first show that: for a symmetric matrix $\mathfrak{A}\in\mathrm{M}_{n+1}(K)$ with $\mathrm{Det}\,\mathfrak{A}=0$, if the cofactor of $\mathfrak{A}$ at the position $(n+1,n+1)$ is 0, then the cofactor of $\mathfrak{A}$ at the position $(n+1,1)$ is also 0. Further, we may assume that *all* Hankel's determinants $\mathrm{H}_r^{(k)}(F)$, $k\in\mathbb{N}$, are 0.)

(e) (Periodic sequences) A sequence $F\in K^{\mathbb{N}}$ is called periodic if there exist a $s\in\mathbb{N}^+$ and $r\in\mathbb{N}$ such that $F(n+s)=F(n)$ for all $n\geq r$. If $r=0$, then we say that F is strong-periodic with period-length s. For a strong-periodic sequence F, if s_0 is the smallest (positive) period-length, then all other period-lengths are multiples of s_0.

1) A sequence $F \in K^N$ is periodic (resectively strong-periodic) if and only if it satisfies a recursion equation $P(\mathrm{s})F = 0$ for some polynomial $P = X^r(X^s - 1)$, $r \in \mathbb{N}$, $s \in \mathbb{N}+$ (respectively $P = X^s - 1$, $s \in \mathbb{N}^+$).

2) A sequence $F \in K^{\mathbb{N}}$ is strong-periodic if and only if the minimal recursion polynomial μ_F of F has the following property: The residue class x of X in the residue algebra $K[X]/K[X]\mu_F = K[x]$ is a unit and its order in the unit group $K[x]^\times$ is positive. Moreover, in this case the order $\operatorname{Ord} x$ is the smallest positive period length of F.

3) Let $K = \mathbb{F}_q$ be a finite field with $q \in \mathbb{N}^+$ elements. Every sequence $F \in \mathbb{F}_q^{\mathbb{N}}$ which satisfies a recursion equation $P(\mathrm{s})F = 0$ with $\in \mathbb{F}_q[X]$, $P \neq 0$, is periodic and moreover, strong-periodic if $P(0) \neq 0$. If $\mu_F(0) \neq 0$ and $\deg \mu_F = m$, then the minimal positive period length $\geq q^m - 1$ and is exactly $= q^m - 1$ if and only if μ_F is a *primitive prime polynomial*.[7] For example, $X^{15} + X + 1$ is a primitive prime polynomial over $\mathbb{F}_2$. A strong-periodic $0-1$-sequence F of the minimal period length $2^{15} - 1$ is defined by $F(n+2) = F(n) + F(n+1)$, $n \in \mathbb{N}$ and arbitrary initial conditions $F(0), F(1), \ldots, F(14) \in \mathbb{F}_2$.

4) Consider the Fibonacci sequence (f_n) with $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$. modulo the prime numbers 11 and 13. Determine their minimal period lengths.

(f) (Formula of Bernoulli) If $F \in \mathbb{C}^{\mathbb{N}}$ is a sequence with minimal recursion polynomial μ_F, then show that $F(n) \neq 0$ for large enough n and

$$\lim_{n \to \infty} \frac{F(n+1)}{F(n)} = a\,,$$

where $a \in \mathbb{C}$ is the *dominante zero* [8] of μ. (For example, if a sequence $F \in \mathbb{C}^{\mathbb{N}}$ satisfies the initial conditions $F(0) = \cdots = F(m-2) = 0$, $F(m-1) = 1$ and the equation $P(\mathrm{s})F = 0$, then $\mu_F = P$. In reasonable cases, the quotients $F(n+1)/F(n)$ give a very fast converging approximation of a. One can easily test this in the case $P := X^2 - 2X - (a-1)$, $a \in \mathbb{R}^+$ or for $P := X^m - X - 1$, $m \geq 2$.)

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[7] A monic prime polynomial $\pi \in \mathbb{F}_q[X]$ is called a primitive prime polynomial if the residue class x of X in the residue field $L := \mathbb{F}_q[X]/\mathbb{F}_q[X]\pi$ is a generating element of the (cyclic) multiplicative group $L\times$ of L. There are exactly $\frac{1}{n}\varphi(q^n - 1)$ primitive polynomials of degree n in $\mathbb{F}_q[X]$, where φ is the Euler's totient function. The primitive polynomials over $\mathbb{F}_2$ of degree ≤ 4 are:$X = 1$, $X^2 + X + 1$, $X^3 + X + 1$, $X^3 + X^2 + 1$, $X^4 + X + 1$, $X^4 + X^3 + 1$. Determine primitive prime polynomials over another finite fields!

[8] We say that a monic polynomial P of degree $m \geq 1$ has the dominante zero $a \in \mathbb{C}$ if $a \neq 0$, $P(a) = 0$ and $\|b| < |a|$ for every zero b of P with $b \neq a$.