1.B Vector Spaces

Before we give formal definition
Consider the following important example.

1.B.1 Example Let K be a field and let n be a natural number. The set of n-tuples of elements from K is the set

$$K^n = \{(a_1, \dots, a_n) \mid a_1, \dots, a_n \in K\}.$$

This set has the natural structure of an abelian group with resp. to the componentwise addition:

$$(a_1, \dots, a_n) + (b_1, \dots, b_n) := (a_1 + b_1, \dots, a_n + b_n).$$

The zero element is $0 = (0, \dots, 0)$ and the negative of $(a_1, \dots, a_n)$ is:

$$-(a_1, \dots, a_n) = (-a_1, \dots, -a_n).$$

Moreover, we also have a natural operation of K on K^n: for every $a \in K$ and every $x = (a_1, \dots, a_n) \in K^n$, the a-times of x is:

$$ax = a(a_1, \dots, a_n) := (aa_1, \dots, aa_n);$$

this is componentwise multiplication by a.
The following rules for calculation are easy to verify (corresponding to the rules for calculation in K)

(1) $a(bx) = (ab)x$ (2) $a(x+y) = ax+by$

(3) $(a+b)x = ax+bx$ (4) $1 \cdot x = x$,

where a, b are arbitrary elements in K and x, y are arbitrary elements in K^n.

The operation of K on K^n considered in the above example is a special case of the following general concept:

<u>1.B.2 Definition</u> Let A and X be sets. By an <u>operation</u> of A on X we mean a map $A \times X \longrightarrow X$.

A binary operation on a set X is an operation of X onto itself. For the pair $(a,x) \in A \times X$ the image under the operation map $A \times X \to X$ is denoted by $ax \in X$. For a fixed $a \in A$, the map $X \longrightarrow X$, $x \mapsto ax$ is called the <u>operation</u> of a on X

In 1.B.1 the space K^n are prototype for vector spaces. More generally, we define

<u>1.B.3 Definition</u> Let K be a field. An (additively written) commutative group $(V,+)$ with an operation of K onto V is called a

K-Vector space or vector space over K

if: for all $a, b \in K$ and all $x, y \in V$ we have:

(1) $a(bx) = (ab)x$.

(2) $a(x+y) = ax + by$.

(3) $(a+b)x = ax + bx$.

(4) $1 \cdot x = x$.

We use the bracket-convention, for example instead of $(ax)+(by)$ we simply write $ax+by$.

If V is a K-vector space, then the elements of V are called vectors and the elements of K are called scalars. The operation of K onto V is called the scalar multiplication. for $a \in K$ and $x \in V$, $ax \in V$ is called the a-times or a-fold x.

With resp. to the addition $(V, +)$ is the additive group of the vector space V.

The zero element $0 = 0_V$ is called the zero vector. If $K = \mathbb{R}$ or $\mathbb{C}$, then V is called real (resp. complex) vector space.

From the distributive laws (2) and (3) in 1.B.3, the following general distributive law follows:

$$\left(\sum_{i\in I} a_i\right)\left(\sum_{j\in J} x_j\right) = \sum_{(i,j)\in I\times J} a_i x_j$$

for arbitrary finite families a_i, $i\in I$, in K and x_j, $j\in J$, in V

For the scalar multiplication the following rules are easy to prove:

(1) $0\cdot x = a\cdot 0 = 0$ (2) $a(-x) = -(ax)$

(3) $(-a)(-x) = ax$ (4) $a(x-y) = ax - ay$

and $(a-b)x = ax - bx$

for all $a, b \in K$ and all $x, y \in V$.

Note that in (1) the zero-element in K and the zero-vector in V are both denoted by 0.

Further, we also have the following:

Cancellation-rule: From $ax = 0$ for $a \in K$ and $x \in V$, it follows that either $a = 0$ or $x = 0$, since if $a \neq 0$, then $x = 1 \cdot x = (a^{-1}a) \cdot x = a^{-1}(ax) = a^{-1} \cdot 0 = 0$.

1.B.4 Example (Function-spaces)
Let K be a field and let I be an arbitrary set. The set K^I of K-valued functions $I \longrightarrow K$ on I is a K-vector space with respect to the addition of functions $f, g \in K^I$:
$(f+g)(i) = f(i) + g(i)$, $i \in I$ and the multiplication of functions $f \in K^I$ with constants $a \in K$: $(af)(i) = a f(i)$, $i \in I$.

For example, $K^{\mathbb{N}}$ is the K-vector space of all (infinite) sequences with values in K. The vector space K^n of 1.B.1 is a special case for $I = \{1, \dots, n\}$. For $I = \{1\}$, $K = K^1$ is a vector space over itself, the scalar multiplication is identical with the multiplication in K.

1.B.5 Example (Restriction of scalars)

Let K be a field and let $K' \subseteq K$ be a subfield of K, i.e. the addition and the multiplication of K' is the restriction of the corresponding binary operations of K. Then K is a K'-vector space, where the scalar multiplication of elements from K' with the elements of K is given by the multiplication in K. For example, $\mathbb{C}$ and $\mathbb{R}$ are vector spaces over $\mathbb{Q}$. The vector space $\mathbb{C}$ over the subfield $\mathbb{R}$ of $\mathbb{C}$ is not other than $\mathbb{R}^2$, see vol 1, §5.A.

More generally, every K-vector space V is also K'-vector space, where the elements $a' \in K' \subseteq K$ operate on V as the given operation of K on V. In this case we say that the K'-vector space structure on V is obtained by the restriction of scalars to K'. In particular, every complex vector space is also a real-vector space. This is often very useful.

1.B.6 Example (Direct Products and Direct sums) Let V_i, $i \in I$, be a family of vector spaces over a field K. The cartesian product $\prod_{i \in I} V_i$ with the componentwise addition and scalar multiplication is clearly a K-vector space. This vector space is called the direct-product of the family V_i, $i \in I$.

The set of I-tuples $(x_i) \in \prod_{i \in I} V_i$ with $x_i = 0$ for almost all $i \in I$ is clearly also a K-vector space, where the addition and scalar multiplication (as in the direct-product) are defined componentwise. This K-vector space is called the direct sum of the family V_i, $i \in I$ and is denoted by $\bigoplus_{i \in I} V_i$ or $\coprod_{i \in I} V_i$.

For a finite set I, the direct product and the direct sum of V_i, $i \in I$, are naturally equal. In the case $I = \{1, \cdots, n\}$, we denote it by

$$V_1 \times \cdots \times V_n = V_1 \oplus \cdots \oplus V_n.$$

If I is arbitrary and $V_i = V$ for all $i \in I$, then $\prod_{i \in I} V_i = V^I$ is the space of all maps from I into V. The direct sum $\bigoplus_{i \in I} V_i$ is then the space of all maps from I into V such that only finitely many $i \in I$ have non-zero image in V. This is denoted by $V^{(I)}$. The important special cases are $K^{(I)}$ and in particular, $K^{(\mathbb{N})}$ the vector space of those sequences with values in K with almost all of its terms are 0.

<u>1.B.7 Remark</u> (<u>Modules</u>) An abelian group V together with an operation

of a ring R on V for which the conditions (1) to (4) of the definition 1.B.3 are satisfied is called an <u>R-module</u>. A non-trivial example of this general situation is an abelian group H with the multiples as an operation of the ring $\mathbb{Z}$ of all integers on H. The constructions given in the above examples are more generally can be done for an arbitrary ring R; this gives more examples of modules. In particular, every R-module is an R'-module by restrictions of scalars to every subring R' of R. Recall that $R' \subseteq R$ is a <u>subring</u> of a ring R if R' is a ring and its both binary operations are restrictions of the corresponding binary operations of R. and the identity element of R' is the identity element of R (this condition is automatically satisfied if R is a field and R' is not the zero ring (proof!), but in general it is not satisfied)