1.C Subspaces

Subspaces are vector spaces their structure is inherited from the vector space they are contained in. For example the direct sums are subspaces of the corresponding direct product.

For a general discussion ~~on~~ the concept of subspaces, we first recall subgroups in a group.

1.C.1 Definition Let G and H be groups with $H \subseteq G$. Then H is called a *subgroup* of G if the binary operation of H is obtained by the restriction of the binary operation of G.

The neutral element e_H of a subgroup H of a group is equal to the neutral element e of G. From $e_H \cdot e_H = e_H = e_H \cdot e$, it follows (by cancelling e_H) $e_H = e$.

Further, the inverse of an element a

in H is necessarily the inverse of a in G (proof!) Further, if $a, b \in H$, the product $ab \in H$. These properties are clearly characterise the subgroups of G. Therefore:

1.C.2 Subgroup-Criterion *Let G be a group with neutral element e. A subset $H \subseteq G$ is a subgroup of G if and only if: (1) $e \in H$ (2) if $a, b \in H$, then ab and $a^{-1} \in H$.*

The condition (2) in 1.C.2 can also be reformulated as: (2′) *if $a, b \in H$, then $ab^{-1} \in H$.*

From 1.C.2 we immediately have:

1.C.3 *Intersection of an arbitrary family of subgroups of a group is again a subgroup.*

From 1.C.3 it follows that: for

every family a_i, $i \in I$, of elements in a group G, there is a smallest subgroup of G containing these elements, namely, the intersection of all subgroups of G containing $\{a_i \mid i \in I\}$ (this family of subgroups is non-empty, since G is one of them). This group is said to be the subgroup of G generated by a_i, $i \in I$ and the family a_i, $i \in I$ is called the generating system of this subgroup.

In the case $I = \{1\}$, $a_1 = a$, this subgroup exactly contain the powers a^n, $n \in \mathbb{Z}$ (resp. the multiples na, $n \in \mathbb{Z}$ in additive notation) of a.

In the additive notation we denote this subgroup by $\mathbb{Z} \cdot a$.

More generally, subgroups generated by single element are called cyclic

<u>Subgroups</u>. A group is called <u>cyclic</u> if itself is generated by a single element.

In an additively written commutative group G, the subgroup generated by the given family a_i, $i \in I$, in G, precisely contains the elements $\sum_{i \in I} n_i \cdot a_i$ with $(n_i)_{i \in I} \in \mathbb{Z}^{(I)}$.
It is therefore the set of all finite sums of integral multiples of a_i, $i \in I$.
We therefore denote this subgroup by $\sum_{i \in I} \mathbb{Z} \cdot a_i$. For $I = \{1, \cdots, n\}$, we also write $\mathbb{Z} \cdot a_1 + \cdots + \mathbb{Z} \cdot a_n$.

<u>1.C.4 Example</u> (<u>Subgroups of $\mathbb{Z}$</u>)
The group $\mathbb{Z} = (\mathbb{Z}, +)$ is cyclic and is generated by 1 as well as -1 (and no other element) all of its subgroups are also cyclic. More precisely:

<u>1.C.5 Theorem</u> For every subgroup H of $\mathbb{Z}$ there exists a unique $n \in \mathbb{N}$ such that $H = \mathbb{Z}\cdot n$.

<u>Proof</u> In the case $H = 0$, $n = 0$. In the case $H \neq 0$, H contains a positive integer; since if $a \in H$, $a \neq 0$, then either a or $-a$ is positive. Let n be the smallest positive natural number contained in H (exists by well-ordering property of $\mathbb{N}$). Then $H = \mathbb{Z}n$. For, since $n \in H$, $\mathbb{Z}n \subseteq H$. Conversely, if $a \in H$ is arbitrary, then by division with remainder $a = qn + r$, $0 \leq r < n$, $q \in \mathbb{Z}$, i.e. $r = a + (-q)n \in H$ and hence $r = 0$ by the choice of n. This proves that $a = qn \in \mathbb{Z}\cdot n$. The uniqueness of n is trivial.

The proof of the following assertion is easy:

1.C.6 Theorem Let $a_1, \dots, a_n \in \mathbb{N}^*$.
Then $\mathbb{Z}a_1 + \dots + \mathbb{Z}a_n = \mathbb{Z}\cdot\gcd(a_1, \dots, a_n)$
and $\mathbb{Z}a_1 \cap \dots \cap \mathbb{Z}a_n = \mathbb{Z}\cdot\mathrm{lcm}(a_1, \dots, a_n)$.

Now let V be a vector space over a field K.

1.C.7 Definition A subset U of V is called a K-sub(vector)-space of V if U is a subgroup of the additive group of the vector space V and the scalar multiplication of V restricted to U, i.e. if $a \in K$ and $x \in U$, then $ax \in U$.

Subvector spaces are therefore themselves vector spaces over the field K. From the subgroup criterion 1.C.2 and the definition, we immediately have the following:

1.C.8 Subspace criterion Let V be a K-vector space. A subset U of V.

is a subspace of V if and only if: (1) $0 \in U$ (2) If $x, y \in U$, then $x+y \in U$ (3) If $a \in K$ and $x \in U$, then $ax \in U$.

Proof Note that if $x \in U$, then $-x = (-1).x \in U$ by (3) and hence we can apply 1-C.2. The conditions (2) and (3) in 1-C.8 clearly can be replaced by: if $x, y \in U$ and if $a, b \in K$, then $ax + by \in U$.

As in the subgroup case we have:

1.C.9 The intersection of an arbitrary family of K-subspaces of a K-vector space V is again a K-subspace.

1.C.10 Examples

(1) Every K-vector space V has two trivial subspaces the zero-subspace $0 = \{0\}$ and the whole vector space V.

(2) The field K as a vector space over itself has exactly two subspaces, namely the trivial ones 0 and K. For, if $U \subseteq K$ is a subspace of K and if $U \neq 0$, then there is $x \in U$, $x \neq 0$ and hence it contains every $y \in K$, since $y = (yx^{-1})x \in U$.

(3) For every element x in a K-vector space V, $Kx := \{ax \mid a \in K\}$ is a subspace of V; this immediately follow from the subspace criterion 1.C.8. Kx is clearly the smallest subspace of V which contain the element x.

(4) Let D be a subset of $\mathbb{R}$ or $\mathbb{C}$. The set of all continuous functions $D \longrightarrow \mathbb{K}$ is a subspace of the vector space $\mathbb{K}^D$ of all $\mathbb{K}$-valued functions on D. This follows from the subspace criterion 1.C.8 and rules for continuous functions. We denote this subspace

by $C_{\mathbb{K}}(D) = C^0_{\mathbb{K}}(D) = C(D)$

(5) Let I be an interval in $\mathbb{R}$ (containing more than one point) and let $n \in \mathbb{N}$. The set of n-times continuously differentiable $\mathbb{K}$-valued functions on I is a subspace of the $\mathbb{K}$-vector space $C_{\mathbb{K}}(I)$. This follows from the subspace criterion 1.C.8 and rules for continuously differentiable functions. We shall denote this $\mathbb{K}$-subspace by $C^n_{\mathbb{K}}(I)$

The subspaces $C^n_{\mathbb{K}}(I)$, $n \in \mathbb{N}$, form an descending chain

$$C^0_{\mathbb{K}}(I) \supsetneq C^1_{\mathbb{K}}(I) \supsetneq \cdots \supsetneq C^n_{\mathbb{K}}(I) \supsetneq C^{n+1}_{\mathbb{K}}(I) \supsetneq$$

where all inclusions are proper. The intersection of all these subspaces is the $\mathbb{K}$-subspace

$$C^{\infty}_{\mathbb{K}}(I) = \bigcap_{n \in \mathbb{N}} C^n_{\mathbb{K}}(I)$$

of infinitely many times differentiable

functions on I. An another subspace of $C^{\infty}_{\mathbb{K}}(I)$ is the space $C^{\omega}_{\mathbb{K}}(I)$ of all $\mathbb{K}$-valued <u>analytic</u> functions on I. The inclusion $C^{\omega}_{\mathbb{K}}(I) \subset C^{\infty}_{\mathbb{K}}(I)$ is also proper. This follows from the existence of a flat-functions, cf. Vol 1, Example 13.C.17.

(6) Let K be a field and D be a subset of K. The set of polynomial functions $D \longrightarrow K$ is a K-subspace of the K-vector space K^D of all maps from D into K. We recall here that a <u>polynomial function</u> is a function of the form

$$t \longmapsto a_0 + a_1 t + \cdots + a_n t^n$$

with fixed coefficients $a_0, \ldots, a_n \in K$.

If D is finite, then every function on D is a polynomial function, since for finitely many pairwise distinct points $t_1, \ldots, t_r \in K$ and given values

$b_1, \dots, b_r \in K$, there is a polynomial function f such that $f(t_i) = b_i$ for all $i = 1, \dots, r$, see Vol 1, § 11.A or § 15.B. <u>If D is infinite then the coefficients $(a_0, \dots, a_n)$ are uniquely determined the functions.</u> This follows from the <u>identity-theorem</u> Vol 1, 11.A.5. (The proof there also works for an arbitrary field, see also 10.A.12)

For infinite subset $D \subseteq K$, the space of polynomial functions on D is denoted by $K[t]$ and we can talk about the <u>degree</u> of a polynomial function $f \in K[t]$. For every $n \in \mathbb{N}$, the polynomial functions of degree $< n$ form a K-subspace which we will denote by $K[t]_n$. It precisely contain <u>polynomial functions</u> from $K[t]$ which

<u>10.A.12</u> <u>Identity</u> <u>Theorem</u> Let F, G be polynomials with coefficients in a field K of degrees $\leq n$. Suppose that for $n+1$ distinct places $t_1, \dots, t_{n+1} \in K$, $F(t_i) = G(t_i)$ for all $i = 1, \dots, n+1$. Then $F = G$.

can be written in the form
$t \longmapsto a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$ with
$a_0, \dots, a_{n-1} \in K$. If I is an interval
in $\mathbb{R}$, then $\mathbb{K}[t]$ is a subspace of
$C^{\omega}_{\mathbb{K}}(I)$.

Unions of subspaces are not in
general subspace (see Exercise 3).
Let U_i, $i \in I$, be subspaces of a
K-vector space V. The smallest
subspace of V which contain all
U_i, $i \in I$, is called the <u>sum</u> of U_i, $i \in I$
and is denoted by $\sum_{i \in I} U_i$. We
will write the elements in this
subspace.

<u>1.C.11 Theorem</u> *Let U_i, $i \in I$, be
a family of subspaces of a K-vector
space V. Then the sum $\sum_{i \in I} U_i$ contains
precisely elements of the form $\sum_{i \in I} x_i$*

$x_i \in U_i$ for all $i \in I$ and $x_i = 0$ for almost all $i \in I$, i.e.

$$\sum_{i\in I} U_i = \Big\{ \sum_{i\in I} x_i \;\Big|\; \begin{array}{l} x_i \in U_i \text{ for all } i \in I \\ \text{and } x_i = 0 \text{ for almost all } i \in I \end{array} \Big\}$$

Proof These elements necessarily belong to $\sum_{i\in I} U_i$. Further, since

$$\sum_{i\in I} x_i + \sum_{i\in I} y_i = \sum_{i\in I} (x_i + y_i), \quad a \sum_{i\in I} x_i = \sum_{i\in I} a x_i$$

for $(x_i), (y_i) \in \bigoplus_{i\in I} U_i$ and $a \in K$,

they form a subspace which contain every subspace U_i, $i \in I$ and hence this subspace is equal to $\sum_{i\in I} U_i$.

In particular, the sum of finitely many subspaces $U_1, \dots, U_n$ of V contains precisely elements of the form $x_1 + \dots + x_n$ with $x_i \in U_i$ for all $i = 1, \dots, n$

Let x_i, $i \in I$, be an arbitrary family of elements of V. The smallest subspace of V which contain all x_i, $i \in I$, i.e. the subspaces $Kx_i = \{ax_i \mid a \in K\}$, $i \in I$, is precisely the sum $\sum_{i \in I} Kx_i$.

This subspace is called the <u>subspace generated by</u> x_i, $i \in I$ and x_i, $i \in I$, is called a <u>generating system</u> of this subspace. By 1-C.11 this subspace contain exactly the elements of the form $\sum_{i \in I} a_i x_i$ with coefficients $a_i \in K$ almost all of which are zero. These elements are called the <u>linear combinations</u> of x_i, $i \in I$. For caculation of those elements we have:

$$\sum_{i \in I} a_i x_i + \sum_{i \in I} b_i x_i = \sum_{i \in I} (a_i + b_i) x_i$$

$$a \sum_{i \in I} a_i x_i = \sum_{i \in I} (a a_i) x_i$$

The linear combinations of finitely many elements $x_1, \ldots, x_n$ have the form

$a_1x_1 + \cdots + a_nx_n$ with $a_1, \ldots, a_n \in K$.

We end this section with the following frequently used lemma:

1.C.12 Lemma *Let V be a vector space over a field K and let $V_1, \ldots, V_n$ be subspaces of V. If K has at least n elements and $V_1, \ldots, V_n$ are all different from V, then the union $V_1 \cup \cdots \cup V_n \subsetneq V$.*

Proof (By induction on n) For $n \le 1$ the assertion is trivial. For the inductive step from $n-1$ to $n \geq 2$, by induction-hypothesis we may assume that there is $x \in V$ with $x \notin V_1 \cup \cdots \cup V_{n-1}$ and $y \in V$ with $y \notin V$

Then the elements $ax+y$, $a \in K$, are pairwise distinct, since for distinct $a, b \in K$, the difference $(ax+y)-(bx+y) = (a-b)x \neq 0$ since $x \neq 0$. By the choice of x, for every $i=1,\dots,n-1$, ~~there is~~ atmost one $a_i \in K$ with $a_i x + y \in V_i$. ~~Mor~~ If $a \in K$ is different from all those a_i, $i=1,\dots,n-1$, then the element $ax+y \notin V_1 \cup \dots \cup V_{n-1}$ (by choice of x and y) and $ax+y \notin V_n$.)

Many times 1-C.12 is used in the following form: If $V_0, V_1, \dots, V_n$ are subspaces of a vector space V over a field K which has at least n elements and if $V_0 \nsubseteq V_i$ for all $i=1,\dots,n$, then $V_0 \nsubseteq V_1 \cup \dots \cup V_n$

For the proof apply 1-C.12 to the vector space V_0 with the proper subspaces $V_1 \cap V_0, \dots, V_n \cap V_0$.