

§2 System of Linear Equations

2.A Gauss-Elimination 2A/1-2A/

2 Linear Equations

2.A Gauss Elimination Process

Let a_1, \dots, a_m and b ^{be} elements of a field K . By a solutions of the linear equation

$$a_1 x_1 + \dots + a_m x_m = b$$

with n variables (or unknowns) x_1, \dots, x_n , we understand all n -tuples $x = (x_1, \dots, x_n) \in K^n$ for which the given eqn. holds. It is very easy to give all solutions of this equation. If all a_1, \dots, a_m are 0, then there is a solution if and only if $b = 0$ and in this case all $x \in K^n$ are solutions. In the other case at least one of a_1, \dots, a_m is non-zero, say a_1 , then clearly all solutions are n -tuples $x = (x_1, \dots, x_n) \in K^n$ and that $x_2, \dots, x_n \in K$ are arbitrary and

$$x_1 = a_1^{-1} b - a_1^{-1} a_2 x_2 - \dots - a_1^{-1} a_m x_m.$$

We now consider a system 2A/2

$$\mathcal{E}: \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

of m linear equations in n unknowns x_1, \dots, x_n with coefficients $a_{ij}, b_j \in K$, $i = 1, \dots, m$; $j = 1, \dots, n$.

Solutions are ^{the} n -tuples $x = (x_1, \dots, x_n) \in K^n$ which satisfy all m equations simultaneously. $L(\mathcal{E})$.

If all b_1, \dots, b_m are 0, then this system is called a homogeneous system of linear equations. In the

general case the system of equations is called inhomogeneous and the

\mathcal{E}_0 system of ^{linear} equations obtained from this system by putting all $b_i = 0$ is called the corresponding homogeneous linear system of equations.

The solutions of ^{both} these systems of equations are related as follows:

2.A.1 Theorem

(1) Let x' be a (special) solution of the inhomogeneous system of equations. The all solutions of this system is obtained by adding ^{to x'} a solution of the corresponding homogeneous system of linear equations, i.e.

$$L(\varepsilon) = x' + L(\varepsilon_0) = \{x' + y \mid y \in L(\varepsilon_0)\}$$

(2) The solutions-set $L(\varepsilon_0)$ of the homogeneous system of linear equations is a K -subspace of K^n .

Proof (1) Let $x' = (x'_1, \dots, x'_n)$. Then

$$\sum_{j=1}^n a_{ij} x'_j = b_i \text{ for all } i=1, \dots, m.$$

For an element $x = (x_1, \dots, x_n) \in K^n$,

$$\text{we have } \sum_{j=1}^n a_{ij} x_j = b_i \text{ for all } i=1,$$

$$\dots, m \iff \sum_{j=1}^n a_{ij} (x_j - x'_j) = 0 \text{ for}$$

all $i=1, \dots, m$, i.e. $y = x - x' \in L(\varepsilon_0)$.

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Therefore to find all solutions of a given system of inhomogeneous linear equations, it is enough to find one solution x' of this inhomogeneous system and a generating system $y^{(1)}, \dots, y^{(k)}$ for the solution-space $L(\xi_0)$ of the corresponding system of homogeneous linear equations.

Then every solution of ξ has the form

$$x' + c_1 y^{(1)} + \dots + c_k y^{(k)}$$

$c_1, \dots, c_k \in K$. The problem is to choose the no. of generators k as small as possible, for this we introduce concepts of the basis resp. dimension of a vector space in the next paragraphs.

Identify an equation $a_1 x_1 + \dots + a_m x_m = b$ with the coefficient tuple

$$(a_1, \dots, a_m, b) \in K^m$$

This allows us to use the operations

of the vectors in K^{n+1} and translate them back into equations. Therefore clearly: The solution-set of a system of linear equations will remain unchanged if we perform the following well-known elementary operations:

- (1) Adding a multiple of an equation into other equation.
- (2) Multiplying ~~an~~ an equation with $a \in K, a \neq 0$.
- (3) Interchanging equations.

The Gauss. elimination is a systematic application of these elementary operations to obtain all solutions algorithmically:

If at least in one of the equations

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

...

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

by interchanging the equations, we may assume that $a_{11} \neq 0$ (this element a_{11} is called Pivot-element) (Pivot = angel in English)

Then multiply the first equation by a_{11}^{-1} and add it to every $i=2, \dots, m$ row ~~to~~ the $(-a_{i1})$ -multiple of

Then all coefficients of x_1 in each equation for $i \geq 2$ are 0.

Now, one repeat this process to the system of equations from 2nd to m -th equation and eliminate the unknown x_1 . If all $a_{i1} = 0$ for $i=1, \dots, m$, then already we can eliminate x_1 . Proceeding this way we arrive at a system of equations which has the following form:

$$1 \leq i_1 < i_2 < \dots < i_r \leq n$$

~~$$x_1 + c_{1,1}x_{i_1+1} + c_{1,2}x_{i_2+1} + \dots$$~~

$$x_{i_1} + c_{1,1}x_{i_1+1} + \dots + c_{1,2}x_{i_2} + c_{1,2+1}x_{i_2+1} + \dots + c_{1,n}x_n = d_1$$

$$x_{i_2} + c_{2,2+1}x_{i_2+1} + \dots + c_{2,n}x_n = d_2$$

$$x_{i_r} + \dots + c_{r,n}x_n = d_r$$

$$0 = d_{r+1}$$

$$0 = d_m$$

This system has a solution if and only if $d_{r+1} = \dots = d_m = 0$. Moreover, in this case the value of for the unknowns x_i with $i \in \{i_1, \dots, i_r\}$ can be freely chosen (in K) and the value for x_{i_1}, \dots, x_{i_r} are then uniquely determined

from the r -th, $(r-1)$ -th, ..., 2-nd, 1-st equations. ^{The calculation in} This last step can be obtained by further use of elementary operations on the coefficients of x_{i_j} in the 1-st till $(j-1)$ -th equations to 0 for $j=2, \dots, r$.

Choose Pivot. element as the $\frac{2A}{8}$
biggest among a_{ij}

2.A.2 Example Find solution-set
of the following system by using Gauss-
elimination

$$2x_1 + x_2 - 2x_3 + 3x_4 = 1$$

$$3x_1 + 2x_2 - x_3 + 2x_4 = 4$$

$$3x_1 + 3x_2 + 3x_3 - 3x_4 = b$$

over \mathbb{Q} .

First Pivot-element $a_{11} = 2$

$$x_1 + \frac{1}{2}x_2 - x_3 + \frac{3}{2}x_4 = \frac{1}{2}$$

$$\frac{1}{2}x_2 + 2x_3 - \frac{5}{2}x_4 = \frac{5}{2}$$

$$\frac{3}{2}x_2 + 6x_3 - \frac{15}{2}x_4 = b - \frac{3}{2}$$

Next Pivot-element $c_{22} = \frac{1}{2}$

$$x_1 + \frac{1}{2}x_2 - x_3 + \frac{3}{2}x_4 = \frac{1}{2}$$

$$\frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 = \frac{5}{2}$$

$$0 = b - 9$$

So this system has solution if and only if

$b=9$. For the calculation of ²⁰¹⁹solution set in this case take the equivalent system of equations (replace R_1 by $R_1 - \frac{1}{2}R_2$)

$$x_1 - 3x_3 + 4x_4 = -2$$

$$x_2 + 4x_3 - 5x_4 = 5$$

Then the solution-set is

$$L = \left\{ (x_1, x_2, x_3, x_4) \mid \begin{array}{l} x_1 = 3x_3 - 4x_4 - 2 \\ x_2 = -4x_3 + 5x_4 + 5 \end{array} \right\}$$

$$= \left\{ (3x_3 - 4x_4 - 2, -4x_3 + 5x_4 + 5, x_3, x_4) \mid \right.$$

$$\left. x_3, x_4 \in \mathbb{Q} \right\}$$

$$= \mathbb{Q}(-2, 5, 0, 0) + L(\mathcal{E}_0), \text{ where}$$

$$L(\mathcal{E}_0) =$$

$$= \mathbb{Q}(3, -4, 1, 0) + \mathbb{Q}(-4, 5, 0, 1).$$

The vectors $(3, -4, 1, 0), (-4, 5, 0, 1) \in \mathbb{Q}^4$ generate the \mathbb{K} -subspace $L(\mathcal{E}_0)$ and the sp-soln. $(-2, 5, 0, 0)$ used to obtain

The representation of $L(E)$ given in
2.A.4 (1).