§3 Bases and Dimension of Vector spaces

3.A Generating Systems, Linear independence, Bases. 3A/1 – 3A/35

3.B Dimension of Vector Spaces 3B/1 –

3.A Generating systems, linear independence, Bases

Let V be a K-vector space

3.A.1 Definition A family v_i, $i \in I$ of elements of V is called a <u>generating system</u> of V if every element of V is a linear combination of v_i, $i \in I$, equivalently, $V = \sum_{i \in I} K v_i$ (= the subspace generated by v_i, $i \in I$.

If v_i, $i \in I$, is a generating system of V, then every element $x \in V$ can be written in the form

$$x = \sum_{i \in I} a_i v_i \,, \quad a_i \in K, \quad \underbrace{a_i = 0}_{\text{for all most by}} \ i \in I$$

The coefficients a_i in these representation are in general <u>not</u> uniquely determined by x.

If $x = \sum_{i\in I} b_i v_i$ is another ~~decomposition~~ representation of x as linear combination of v_i, $i \in I$, then

$$0 = x - x = \sum_{i\in I}(a_i - b_i)v_i = \sum_{i\in I} c_i v_i$$

is a representation of 0 with coefficients $c_i := a_i - b_i$, $i \in I$.

If $c_i \neq 0$ for all $i \in I$, i.e. $a_i \neq b_i$ for all $i \in I$, then this is a <u>non-trivial representation</u> of 0, i.e. a representation of 0 with coefficients in K not all zero. Conversely, from every such non-trivial representation of 0, we get another representation of x as linear combination of the v_i, $i \in I$.

Therefore the problem of uniqueness of such a representation reduces to the question of the "linear independence" of the v_i, $i \in I$, in the sense of the following definition:

3.A.2 Definition A family $v_i, i \in I$, of elements in V is called *linear independent* (over K) if the following holds:

If $(a_i)_{i\in I} \in K^{(I)}$, and $\sum_{i\in I} a_i v_i = 0$

then $a_i = 0$ for all $i \in I$.

The linear independence of the v_i, $i \in I$ means: A linear combination of the v_i, $i\in I$, is 0 if and only if (not only almost all, but) all its coefficients are 0.

The v_i, $i \in I$, are *linearly dependent*, i.e. not linearly independent, if there exists an I-tuple $(a_i)_{i\in I} \in K^{(I)}$ such that $\sum_{i\in I} a_i v_i = 0$, but at least for one $i_0 \in I$, $a_{i_0} \neq 0$. Then

$$v_{i_0} = a_{i_0}^{-1} \cdot \sum_{\substack{i\in I\\ i\neq i_0}} a_i v_i = \sum_{\substack{i \in I \\ i \neq i_0}} (-a_{i_0}^{-1} \cdot a_i) \cdot v_i$$

$\left(\begin{matrix} i\in I \\ i \neq i_0 \end{matrix}\right) = i \neq i_0.$

is a linear combination of the remaining v_i, $i \in I$, $i \neq i_0$.

Therefore, a system of vectors is linearly ~~inde~~dependent if and only if no vector in this system can be represented as a linear combination of the remaining vectors.

In particular, in a linearly independent family v_i, $i \in I$, $v_i \neq v_j$ for all $i, j \in I$, $i \neq j$.

A family v_i, $i \in I$, is linearly independent if and only if every finite subfamily v_j, $j \in J$, $J \subseteq I$ finite is linearly independent.

By the remark before def. 3.A-2:

If the system v_i, $i \in I$ of vectors in V is linearly independent and if $x = \sum_{i \in I} a_i x_i \in V$ is a linear combination of the v_i, $i \in I$, then the

I-tuple $(a_i) \in K^{(I)}$ is uniquely determined by x.

3.A.3 Definition A family v_i, $i \in I$, of elements of V is called a basis of V if v_i, $i \in I$, is a linearly independent generating system of V.

If v_i, $i \in I$, is a basis of V, then every vector $x \in V$ ~~has~~ is a linear combination $x = \sum_{i \in I} a_i v_i$, $(a_i) \in K^{(I)}$, of v_i, $i \in I$ and the coefficients a_i, $i \in I$, are uniquely determined by x. The a_i, $i \in I$, are called coordinates or components or coefficients of x with respect to the basis v_i, $i \in I$, and the functions: for $i \in I$,

$$v_i^*: V \longrightarrow K, \quad x \longmapsto v_i^*(x) := a_i$$

are called the coordinate-functions on V with resp. to the basis v_i, $i \in I$.

Therefore by definition: for all $x \in V$,

$$x = \sum_{i \in I} a_i v_i = \sum_{i \in I} v_i^*(x)\, v_i$$

From the computation-rules for the linear combinations, we have the following computation-rules for the coordinate functions v_i^*, $i \in I$:

For all $x, y \in V$ and all $a \in K$

$$v_i^*(x+y) = v_i^*(x) + v_i^*(y) \quad \text{and}$$
$$v_i^*(ax) = a \cdot v_i^*(x).$$

From the earlier definitions we immediately have:

<u>3.A.4 Theorem</u> Let V be a vector space over K and let v_i, $i \in I$ be a family of elements in V. Let $f : K^{(I)} \longrightarrow V$ be the map defined by $(a_i)_{i \in I} \longmapsto \sum_{i \in I} a_i v_i$. Then

(1) v_i, $i \in I$, is a generating system of V $\iff$ f is surjective.

(2) v_i, $i \in I$, is linearly independent $\iff$ f is injective.

(3) v_i, $i \in I$, is a basis of V $\iff$ f is bijective.

3.A.5 Example (Standard bases)

Let K be a field, I an indexed set, $V = K^{(I)}$ = the I-tuples $(a_i)_{i \in I}$ with elements in K with almost all $a_i = 0$.

For $i \in I$, $e_i := (\delta_{ji})_{j \in I} : I \longrightarrow K$, $i \longmapsto 1$, $j \longmapsto 0,\ j \neq i$

Then $e_i \in K^{(I)}$, $i \in I$, and they form a basis of $K^{(I)}$ – called the standard basis of $K^{(I)}$.

$$(a_i)_{i \in I} = \sum_{i \in I} a_i e_i \quad \text{for } (a_i)_{i \in I} \in K^{(I)}.$$

The coordinate functions e_i^* are the canonical projections $(a_j) \longmapsto a_i$, $i \in I$

In particular, for $n \in \mathbb{N}^*$, $I = \{1, 2, \cdots, n\}$, $K^I = K^{(I)} = \underbrace{K \times \cdots \times K}_{n\text{-times}}$
$= K^n$

$e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $e_n = (0, \cdots, 0, 1)$ is a standard basis of K^n with coordinate functions: $e_i^* : K^n \longrightarrow K$, $(a_1, \cdots, a_n) \longmapsto a_i$, $i = 1, \cdots, n$.

The standard basis of $K = K^1$ is 1.

<u>3.A.6 Example</u> If v_i, $i \in I$, is a linearly independent family of elements in a vector space V over K, then v_i, $i \in I$, is a basis of the K-subspace $W = \sum_{i \in I} K v_i$ of V generated by the family v_i, $i \in I$.

<u>3.A.7 Example</u> (<u>Polynomial functions</u>) Let $D \subseteq K$ be a subset with at-

least n elements, $n \in \mathbb{N}$. Then the power functions $: D \to K$,

$$1 = t^0, t^2, \dots, t^{n-1}$$

which are elements of $K^D :=$ the vector space of all K-valued functions on D, are linearly independent over K: For, if $a_0 + a_1 t + \dots + a_{n-1} t^{n-1} = 0$, $a_0, a_1, \dots, a_{n-1} \in K$, i.e. the function $p : D \to K$, $t \mapsto p(t) := a_0 + a_1 t + \dots + a_{n-1} t^{n-1} = 0$ on D and hence p has at least n zeros, but $\deg p \leq n-1$. Therefore $p(t) = 0$, i.e. $a_0 = a_1 = \dots = a_{n-1} = 0$.

In particular, the power-functions t^m, $m = 0, \dots, n-1$, is a basis of the space $K[t]_n$ of the polynomial functions of degree $< n$ on D.

If D has infinitely many elements, then the power-functions t^m, $m \in \mathbb{N}$, are linearly independent and hence form a basis of the space $K[t]$ ($\subseteq K^D$) of all polynomial functions on D.

(*) 3.A.8 Example (Rational functions)

Let $D \subseteq \mathbb{C}$ be an infinite subset. By the theorem on division with remainder and the theorem on the partial fractions (11.B.1 and 11.B.2 or 10.A.2 of Vol 1) and Example 3.B.8 the rational functions

$$t^n,\ n \in \mathbb{N}, \text{ and } \frac{1}{(t-a)^m},\ m \in \mathbb{N}^*,\ a \in \mathbb{C} \setminus D$$

together form a basis of the $\mathbb{C}$-vector space of the rational functions on D, see also Example 3.B.8.

— A corresponding assertion also holds for an arbitrary "algebraically closed" field K, instead of $\mathbb{C}$.

3.A.9 Example The $\mathbb{R}$-valued functions (on $\mathbb{R}$) $f_a : \mathbb{R} \longrightarrow \mathbb{R}$, $t \mapsto |t-a|$, $a \in \mathbb{R}$, are linearly independent elements in the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$ of all $\mathbb{R}$-valued functions on $\mathbb{R}$. (See Exercise 17a) of 12.A, Vol 1)

Simple proof: Let $n > 0$ and $c_1, \ldots, c_n \in \mathbb{R}$, $\sum_{i=1}^{n} c_i f_{a_i} = 0$, i.e. $\sum_{i=1}^{n} c_i |t-a_i| = 0$ for all $t \in \mathbb{R}$. To prove $c_1 = \cdots = c_n = 0$. We may assume $a_1 < a_2 < \cdots < a_n$. Then $c_1 |t-a_1| = -\sum_{i=2}^{n} c_i |t-a_i|$ for all $t \in \mathbb{R}$. On the interval $(-\infty, a_2)$ LHS is $c_1|t-a_1| = c_1(t-a_1)$ is a polynomial function of degree ≤ 1. The RHS on $(-\infty, a_2)$ is always $-$ve. So the only possibility is $c_1 = 0$. Continuing this way conclude also that $c_2 = \cdots = c_n = 0$

(*) <u>3.A.10 Example</u> Let $D \subseteq \mathbb{C}$ and for $\alpha \in \mathbb{C}$, let $e^{\alpha t}$ be the function $\mathbb{C}^D \to \mathbb{C}$, $t \mapsto e^{\alpha t}$

$e^{\alpha t} \in \mathbb{C}^D$. Suppose that D has a limit point in $\mathbb{C}$. Then the family $e^{\alpha t}$, $\alpha \in \mathbb{C}$ is linearly independent in the $\mathbb{C}$-vector space $\mathbb{C}^D$ of $\mathbb{C}$-valued functions on D.

<u>Proof</u> For this one need to use <u>identity theorem</u>:

If $\sum_{\nu=1}^{n} a_\nu e^{\alpha_\nu z} = 0$ for a set z of complex numbers which has a limit pt. then $\sum_{\nu=1}^{n} a_\nu e^{\alpha_\nu z} = 0$ for all $z \in \mathbb{C}$ (by the identity thm) and similarly $\sum_{\nu=1}^{n} a_\nu \alpha_\nu e^{\alpha_\nu z} = 0$ and hence $\sum_{\nu=2}^{n} a_\nu (\alpha_1 - \alpha_\nu) e^{\alpha_\nu z} = 0$. Now, use induction on n

Similarly, the functions t^{α}, $\alpha \in \mathbb{C}$, ~~defined~~ on every subset $D \subseteq \mathbb{C} \setminus \mathbb{R}_-$ which has a limit pt in $\mathbb{C} \setminus \mathbb{R}_-$, are linearly independent in the $\mathbb{C}$-vector-space $\mathbb{C}^D$.

Proof. If $\sum_{\nu=1}^{n} a_\nu t^{\alpha_\nu} = 0$ for a set z of complex numbers in $\mathbb{C} \setminus \mathbb{R}_-$ which has a limit pt in $\mathbb{C} \setminus \mathbb{R}_-$. Then prove (as in the ex. above) that $a_1 = \cdots = a_n = 0$

(*) 3.A.12 Example (Quasi-polynomials)

The functions $t^n \cdot e^{\alpha t}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$ are linearly independent in the space $\mathbb{C}^D$ of $\mathbb{C}$-valued function on a subset $D \subseteq \mathbb{C}$ which has a limit point in $\mathbb{C}$. The subspace generated by these functions is called the space of quasi-polynomials. The quasi-polynomials are the solutions of the linear differential equations with constant coefficients. $P(D)y = 0$, $P \in \mathbb{C}[X] \setminus \{0\}$, (See Vol. 1. 19.D. and also Theorem 11.E.3)

3.A.12 Example ($\mathbb{R}$-bases of $\mathbb{C}$-vector spaces) Let V be a $\mathbb{C}$-complex vector space. Then V is also a $\mathbb{R}$-vector space in a natural way. If v_j, $j \in J$ is a $\mathbb{C}$-basis of V, then v_j, iv_j, $j \in J$, $i = \sqrt{-1}$, together is a $\mathbb{R}$-basis of V

Proof These vectors is a $\mathbb{R}$-generating system of V: for $x \in V$, there are complex numbers z_j, $j \in J$ with $x = \sum_{j \in J} z_j v_j$

Then $$x = \sum_{j \in J} (\operatorname{Re} z_j + i \operatorname{Im} z_j) v_j$$

$$= \sum_{j \in J} (\operatorname{Re} z_j) v_j + \sum_{j \in J} (\operatorname{Im} z_j) iv_j$$

is a representation of x as a linear

combination ~~of~~ in the v_j, iv_j, $j\in J$ with coefficients $\operatorname{Re} z_j$, $\operatorname{Im} z_j \in \mathbb{R}$.

Further, if $0 = \sum_{j\in J} a_j v_j + \sum_{j\in J} b_j (iv_j)$ with real coefficients $a_j, b_j \in \mathbb{R}$, then $0 = \sum_{j\in J} (a_j + ib_j) v_j$. Now, since v_j, $j\in J$, are linearly independent over $\mathbb{C}$, it follows that $a_j + ib_j = 0$ for all $j\in J$ i.e. $a_j = 0$ and $b_j = 0$ for all $j \in J$.

As a simple example, we note that from the $\mathbb{C}$-basis 1 of $\mathbb{C}$, we obtain the $\mathbb{R}$-basis $1, i$ of $\mathbb{C}$. More generally, from the $\mathbb{C}$-standard basis $e_1, \dots, e_n \in \mathbb{C}^n$, we get the $\mathbb{R}$-basis

$(1,0,\dots,0)$, $(i,0,\dots,0)$, $\dots$, $(0,\dots,0,1)$, $(0,\dots,0,i)$

of $\mathbb{C}^n$ with $2n$ elements.

3.A.13 Definition Let v_i, $i \in I$, be a family of vectors in a K-vector space V.

(1) v_i, $i \in I$, is called maximal linearly independent in V if v_i, $i \in I$, is linearly independent and the family v_i, $i \in I \cup \{i_0\}$ with $i_0 \notin I$ is linearly dependent for every $v_{i_0} \in V$.

(2) v_i, $i \in I$, is called a minimal generating system for V if v_i, $i \in I$, is a generating system for V and if v_i, $i \in I \setminus \{i_0\}$ is not a generating system for every $i_0 \in I$.

3.A.14 Theorem *Let v_i, $i \in I$, be a family of vectors in a K-vector space V. Then the following statements are equivalent:*

(1) v_i, $i \in I$, is a basis of V.

(2) v_i, $i \in I$, is a maximal linearly independent in V.

(3) v_i, $i \in I$, is a minimal generating system for V.

<u>Proof</u> <u>(1) $\Rightarrow$ (2)</u>: Extend v_i, $i \in I$, by another element $v_{i_0} \in V$, $i_0 \notin I$. Since v_i, $i \in I$, is a basis of V, v_{i_0} is a linear combination of v_i, $i \in I$, i.e. $v_{i_0} = \sum_{i \in I} a_i v_i$, $(a_i) \in K^{(I)}$. But then v_i, $i \in I \cup \{i_0\}$ is not linearly independent

<u>(2) $\Rightarrow$ (1)</u>: $\overset{\text{ons}}{\vee}$ To show that v_i, $i \in I$, is a generating system for V. Let $x \in V$. Then the family v_i, $i \in I$, x, is linearly independent. Therefore there is a linear combination $0 = \sum_{i \in I} a_i v_i + a x$ with $(a_i) \in K^{(I)}$, $a \in K$ and not all coefficients are zero. If $a = 0$, then we get a contradiction to the linear independence of v_i, $i \in I$. Therefore $a \neq 0$ and $x = \sum_{i \in I} (-a^{-1} a_i) \cdot v_i$ is a linear combination of the v_i, $i \in I$.

(1) $\Longrightarrow$ (3): Let $i_0 \in I$. If v_i, $i \in I \setminus \{i_0\}$ is also a generating system for V, then v_{i_0} is a linear combination of v_i, $i \in I \setminus \{i_0\}$, i.e. $v_{i_0} = \sum_{i \in I \setminus \{i_0\}} a_i v_i$ and hence v_i, $i \in I$, is linearly dependent.

(3) $\Longrightarrow$ (1): If v_i, $i \in I$, is linearly dependent, then there exists v_{i_0}, $i_0 \in I$, which is a linear combination of v_i, $i \in I \setminus \{i_0\}$. Therefore v_i, $i \in I \setminus \{i_0\}$, is a generating system for V and so v_i, $i \in I$, is not a minimal generating system for V.

3.A.15 Theorem Let v_i, $i \in I$, be a countable generating system for a K-vector space V. Then this system contains a basis of V. In particular, every vector space with a countable generating system also has a countable basis.

Proof We may assume that $I \subseteq \mathbb{N}$.

Let $J := \{ i \in I \mid v_i \notin \sum_{\substack{j \in I \\ j < i}} K v_j \}$.

Then v_j, $j \in J$, is a basis of V.

These vectors generate V: Suppose on the contrary that the subspace generated by v_j $j \in J$ does not contain all the vectors v_i, $i \in I$. Then choose a minimal element $i_0 \in I$ with $v_{i_0} \notin \sum_{j \in J} K v_j$ (exists, since $I \subseteq \mathbb{N}$)

Then $i_0 \notin J$ but this is a contradiction the definition of J.

The v_j, $j \in J$, are linearly independent:

If $\sum_{j \in J} a_j v_j = 0$ with at least one $a_{j_0} \neq 0$, $j_0 \in J$, then ~~last~~ let j_0 be maximal with this property (there are only finitely many non-zero, so maximal exists!). Then $v_{j_0} \in \sum_{j < j_0} K v_j$ and hence $j_0 \notin J$ by def. of J, a contradiction.

Corollary Let $v_1, \dots, v_n$ be a finite generating system for V. Then V has a basis.

Theorem 3.A.15 can easily be generalised to a basis-~~extension~~ supplement theorem:

3.A.16 Theorem Let v_i, $i \in I$, be a generating system of the K-vector space V. Suppose that the subfamily v_r, $r \in I_0$ with $I_0 \subseteq I$ is linearly independent and $I \setminus I_0$ is countable. Then there exists a subset $J \subseteq I \setminus I_0$ such that v_s, $s \in I_0 \cup J$, is a basis of V.

Proof We may assume $I \setminus I_0 \subseteq \mathbb{N}$.
~~Choose~~ Let $J := \{ i \in I \setminus I_0 \mid v_i \notin \sum_{r \in I_0} K v_r + \sum_{\substack{j \in I \setminus I_0 \\ j < i}} K v_j \}$

3.A.17 Remark (Bases for arbitrary vector spaces — Well-ordering theorem}. Zorn's Lemma)

If $x_0, x_1, x_2, \cdots$ is a sequence of vectors in the K-vector space V which generate V, then by the method of proof of 3.A.15, we obtain a basis of V. For this one remove all "unnecessary" x_i from this sequence, namely, all the x_i who already belong to the subspace generated by the earlier vectors $x_0, \ldots, x_{i-1}$.

As in 3.A.15 one can always construct ~~(obtain)~~ a basis from an arbitrary generating system v_i, $i \in I$, of V if the index set I is so-called Well-ordered, i.e. a total order with the following property: Every non-empty subset of I has a smallest element. It is well-known Well-ordering Theorem which says that every set has such a well-ordering. Cantor considered even more natural assertion. A first proof was given by by Zermelo with the help of the following well-known Axiom of choice: If A_j, $j \in J$, is a non-empty (i.e. $J \neq \phi$) family of non-empty subsets, then the cross-product $\prod_{j \in J} A_j$ is also non-empty.

(equivalently, there is a map (choice map) $f : J \to \bigcup_{j \in J} A_j$ such that $f(j) \in A_j$ for all $j \in J$)

Existence of choice function:

Therefore from the well-ordering theorem we immediately have:

3.A.18 Theorem (Hamel) Every vector space has a basis.

Hamel proved only the existence of bases for the $\mathbb{Q}$-vector space $\mathbb{R}$; such bases are called Hamel's bases

Hamel basis is not known explicitly.

More generally, 3.A.16 also holds without the hypothesis that $I \setminus I_0$ is countable. We shall cite this as the general basis-supplement theorem if we state it without countability in the hypothesis

The proof of 3.A.18 and the just mentioned generalisation of theorem 3.A.16 can be proved by using a well-order on the index set I to the generating system $v_i, i \in I$,

A natural proof is obtained by using Well-known Zorn's Lemma

3.A.19 Zorn's lemma Every inductively ordered set M has (at least) a maximal element.

An ordered set $(M, \leq)$ is called inductively ordered if every chain in M, (totally ordered subset of M) $N \subseteq M$ has an upper bound in M, i.e. an element $x_S \in M$ with $x \leq x_S$ for all $x \in N$. In particular, M is then non-empty since M contains an upper bound for the empty chain $N := \phi \subseteq M$.

The proof of Zorn's lemma essentially uses the Axiom of choice. For this proof we refer to SS, vol.1, Anhang 1.A

We now prove once again 3.A.18 as an application of Zorn's lemma

For this let $v_i, i \in I$, be a generating system of the K-vector space V and

Let $\mathcal{K}$ be the set of those subsets $J \subseteq I$ for which the sub-family $v_j, j \in J$ is linearly independent, i.e.

$$\mathcal{K} := \{ J \subseteq I \mid v_j, j \in J \text{ is linearly independent} \}$$

Then $\mathcal{K}$ is inductively ordered with resp. to the natural inclusion. For, if $\mathcal{L} \subseteq \mathcal{K}$ is a totally ordered subset of $\mathcal{K}$, then $J_{\mathcal{L}} := \bigcup_{J \in \mathcal{L}} J$ belongs to $\mathcal{K}$ (for $\mathcal{L} = \phi$, $J_{\mathcal{L}} = \Phi \in \mathcal{K}$), since if $j_1, \ldots, j_n$ are finitely many distinct elements in $J_{\mathcal{L}}$, $j_\nu \in J_\nu$ for $\nu = 1, \ldots, n$, then $j_1, \ldots, j_n \in J_{\nu_0}$, where J_{ν_0} is a biggest element of $\{J_1, \ldots, J_n\}$, which exists, since $\mathcal{L}$ is totally ordered. In particular, $v_{j_1}, \ldots, v_{j_n}$ are linearly independent. Therefore we can apply Zorn's Lemma and hence there exists a maximal element $J_0 \in \mathfrak{K}$. Therefore clearly x_j, $j \in J_0$, is a maximal linearly independent system contained in V_i, $i \in I$ in V and hence by 3.A.14 is a basis of V (Exercise!). Exactly similar way one can prove the basis supplement theorem 3.A.16 without hypothesis that $I \setminus I_0$ is countable.

(1) It follows from 3.A.15 that:

<u>Exercise 7 of 3A</u>: If a vector space V has a finite (resp. countable) generating system, then every generating system of V contains a finite (resp. countable) generating system.

<u>Proof</u> (a) Suppose $y_1, \ldots, y_n \in V$ is a finite generating system for V and $x_i, i \in I$, be an arbitrary generating system of V. Then $V = Ky_1 + \cdots + Ky_n = \sum_{i \in I} Kx_i$ and for every $j = 1, \ldots, n$,

$$y_j = \sum_{i \in E(j)} a_{ij} x_i \, , \text{ where } E(j) \text{ is a}$$

finite subset of I and $a_{ij} \in K$ for all $i \in E(j)$. Then $E := \bigcup_{j=1}^{n} E(j)$ is a finite subset of I and the subspace generated by $x_i, i \in E$ contains all $y_1, \ldots, y_n$ and hence is equal to V.

(b) As an extension of (v) above we note the following important theorem:

Let $Y \subseteq V$ be an infinite generating system for V. Then an arbitrary system x_i, $i \in I$, of V contain a generating system x_j, $j \in J$, $J \subseteq I$, with $|J| \leq |Y|$.

Proof for every $y \in Y$, there exists a finite subset $E(y) \subseteq I$ with $y \in \sum_{i \in E(y)} A x_i$. Then x_j, $j \in J := \bigcup_{y \in Y} E(y)$

is a generating system for V, and $|J| \leq |Y|$ (this need Y is infinite)

But, if Y is countable, then J is a countable union of countable (finite) sets and hence J is countable and in particular, $|J| \leq |Y|$. This proves the assertion that if there is (infinite) countable generating system for V, then every generating system of V contains a countable generating system.

(Exercise 8)

(2) Let V be a K-vector space which has a countable infinite basis. Then every basis of V is also countable infinite.

Proof Let v_n, $n \in \mathbb{N}$ be a countable infinite basis of V and let x_i, $i \in I$ be an arbitrary basis of V. Then by (1) the generating system x_i, $i \in I$, must contain a countable generating system for V. But since x_i, $i \in I$, is linearly independent, every proper subfamily x_j, $j \in J \subsetneq I$ cannot be generating system for V. This proves that x_i, $i \in I$, itself must be countable, i.e. I is countable.

(Exercise 9)

(3) Let V be a vector space over K. Suppose that there is an uncountable linearly independent system x_i, $i \in I_0$, in V. Then every system of generators of V is also uncountable.

Proof Suppose that there

is a countable system v_n, $n \in \mathbb{N}$ of generators for V. Let $I := I_0 \cup \mathbb{N}$ and let $v_i := x_i$ for $i \in I_0$. Then v_i, $i \in I$ is also a generating system for V. Now, since x_i, $i \in I_0$, is linearly independent and $I \setminus I_0 = \mathbb{N}$ is countable, by 3.A.16 there exists a subset $J \subseteq I \setminus I_0 = \mathbb{N}$ such that v_j, $j \in I_0 \cup J$ is a basis of V.

This proves that the family x_i, $i \in I_0$, is extended to a basis of V and this basis of V is uncountable. But by 3.A.15 V also has a countable basis, since there is a countable generating system v_n, $n \in \mathbb{N}$ for V. This is a contradiction to (2) above.

(4) (Exercise 10) Let K be a field which has only countably many elements

Then every K-vector space V which has a countable system of generators has countably many elements, i.e. V is countable set.

Proof By 3.A.15 V has a countable basis v_n, $n \in \mathbb{N}$. Then the map $K^{(\mathbb{N})} \longrightarrow V$, $(a_n) \mapsto \sum a_n v_n$ is bijective. Therefore it is enough to show that $K^{(\mathbb{N})}$ is countable, if K is countable.

For, this first note that $\mathfrak{P}_f(K) \cong \mathfrak{P}_f(\mathbb{N})$ is countable and the map

$$K^{(\mathbb{N})} \longrightarrow \mathfrak{P}_f(K), \quad (a_n)_{n \in \mathbb{N}} \mapsto \{a_n\}$$

is injective. Therefore $K^{(\mathbb{N})}$ is a subset of a countable set $\mathfrak{P}_f(K)$ and hence countable.

(5) (Exercise 11) Let $v_i, i \in I$, be a generating system of the K-vector space V. Show that every maximal linearly independent subsystem of $v_i, i \in I$, is a basis of V.

Proof Let $J \subseteq I$ be such that $v_j, j \in J$, is maximal linearly independent subsystem of $v_i, i \in I$. Then for every $i \in I \setminus J$, the sub family $v_j, j \in J, v_i$, is not linearly independent, i.e. there is a linear combination $\sum_{j \in J} a_j v_j + b v_i = 0$

$a_j = 0$ for almost all $j \in J$, $a_j, b_i \in K$, and not all a_j, b are zero.

Then $b \neq 0$, since $v_j, j \in J$, is linearly independent and hence

$$v_i = -b^{-1}\Big(\sum_{j \in J} a_j v_j\Big) \in \sum_{j \in J} K v_j$$

This proves that $v_i \in \sum_{j \in J} K v_j$ for all $i \in I \setminus J$

and hence v_j, $j \in J$ is a linearly independent generating system for V, i.e. it is a basis of V.

(6) Remark Using Zorn's lemma and the assertion (5) one can prove the existence of basis in arbitrary K-vector spaces. This proves the basis-supplement theorem for arbitrary vector spaces:

Let v_i, $i \in I$, be a generating system for V and let

$$\mathfrak{J} := \{ J \mid J \subseteq I,\ v_j, j \in J \text{ linearly independent} \}.$$

Note that $\mathfrak{J} \neq \phi$, since $\{i\} \in \mathfrak{J}$ for $i \in I$ with $v_i \neq 0$. $\mathfrak{J}$ is ordered by the natural inclusion. To show that $\mathfrak{J}$ is inductively ordered, let $\mathcal{C} \subseteq \mathfrak{J}$ be a chain (totally ordered) subset. Then $\tilde{J} := \bigcup\limits_{J \in \mathcal{C}} J$ is an upper bound for $\mathcal{C}$ and $\tilde{J} \in \mathcal{F}$. For, if v_j, $j \in \tilde{J}$ is not linearly independent, then there are finitely many $j_1, \dots, j_n \in \tilde{J}$ such that $v_{j_1}, \dots, v_{j_n}$ are not linearly independent. But $j_1, \dots, j_n \in J$ for some $J \in \mathcal{C}$, since $\mathcal{C}$ is a chain. Then $\{v_{j_1}, \dots, v_{j_n}\} \subseteq \{v_j \mid j \in J\}$ which is linearly independent, a contradiction. This proves that $\tilde{J} \in \mathcal{F}$ and $\tilde{J}$ is clearly an upper bound for $\mathcal{C}$ in $\mathcal{F}$.

Now we can apply Zorn's lemma to $(\mathcal{F}, \subseteq)$ to conclude that $\mathcal{F}$ has maximal element $J \subseteq I$. This means that the sub-family v_j, $j \in J$, is maximal linearly independent sub-family of v_i, $i \in I$. By (5) v_j, $j \in J$, is a basis of V.

(7) Let V be a K-vector space and let v_i, $i \in I$, be a generating system for V. Suppose that the subsystem v_j, $j \in J$; $J \subseteq I$, is linearly ~~dep~~ independent. Then there exists a subset ~~I~~ L with $J \subseteq L \subseteq I$ such that v_ℓ, $\ell \in L$, is a basis of V.

<u>Proof</u> Similar to that of (6).
Apply Zorn's lemma to the ordered set with resp. to natural inclusion

$$\mathfrak{J} := \{ L \mid J \subseteq L \subseteq I,\ v_\ell,\ \ell \in L, \text{ is linearly independent} \}.$$

Then $J \in \mathfrak{J}$ and check that $\mathfrak{J}$ is inductively ordered and hence by Zorn's lemma $\mathfrak{J}$ has a maximal element L. By (5) v_ℓ, $\ell \in L$ is a basis of V.

<u>Therefore</u>: Every linearly independent system in V can be extended to a basis of V.