3.B Dimension of Vector spaces

In this section we shall prove that all bases of a vector space V over a field K have the same number of elements.

First we restrict ourselves to vector space which has finite basis.

3.B.1 Exchange Lemma Let V be a vector space with a basis $v_1, \dots, v_n$. Further, let $w = a_1 v_1 + \dots + a_k v_k + \dots + a_n v_n \in V$. If $a_k \neq 0$, then $v_1, \dots, v_{k-1}, w, v_{k+1}, \dots, v_n$ is a basis of V.

Proof (1) First we show that $v_1, \dots, v_{k-1}, w, v_{k+1}, \dots, v_n$ are linearly independent. For this suppose that

$$b_1 v_1 + \dots + b_{k-1} v_{k-1} + bw + b_{k+1} v_{k+1} + \dots + b_n v_n = 0$$

Then by substituting for w we get:

$$(b_1 + ba_1) v_1 + \dots + (b_{k-1} + ba_{k-1}) v_{k-1} +$$

$$ba_k v_k + (b_{k+1} + ba_{k+1})v_{k+1} + \cdots + (b_n + ba_n)v_n = 0.$$

Since $v_1, \ldots, v_n$ are linearly independent, $b_1 + ba_1 = \cdots = b_{k-1} + ba_{k-1} = ba_k = b_{k+1} + ba_{k+1} = \cdots$ $b_n + ba_n = 0$. Now, since $a_k \neq 0$, it follows from $ba_k = 0$ that $b = 0$ and finally then $b_i = 0$ for all $i \neq k$.

(2) We shall show that $v_1, \ldots, v_{k-1}, w, v_{k+1}, \ldots, v_n$ is a generating system for V. Since $v_1, \ldots, v_n$ generates V, it is enough to show that v_k is a linear combination of $v_1, \ldots, v_{k-1}, w, v_{k+1}, \ldots, v_n$. This is clear from the equation:

$$v_k = -a_k^{-1} a_1 v_1 - \cdots - a_k^{-1} a_{k-1} v_{k-1} + a_k^{-1} w - a_k^{-1} a_{k+1} v_{k+1} - \cdots - a_k^{-1} a_n v_n.$$

The following important theorem is a stronger version of 3.A.16 for vector

Spaces with finite basis

3.B.2 Steinitz's exchange Theorem

Let V be a K-vector space, $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_m$ ~~ar~~ be linearly independent elements in V. Then $m \leq n$ and there are $n-m$ elements from $v_1, \dots, v_n$ which together with $w_1, \dots, w_m$ form a basis of V.

Proof We use induction on m. The case $m=0$ is trivial. For the inductive step from $m-1$ to m, we may assume that the elements $w_1, \dots, w_{m-1}$ are extended to a basis of V by adding some $n-m+1$ elements from $v_1, \dots, v_n$. By renumbering we may further assume that $w_1, \dots, w_{m-1}, v_m, \dots, v_n$ is a basis of V. Therefore w_m ~~is~~ has a linear combination ~~there is a~~ ~~representation~~:

$$w_m = a_1 w_1 + \dots + a_{m-1} w_{m-1} + a_m v_m + \dots + a_n v_n$$

Since $w_1, \dots, w_{m-1}, w_m$ are linearly independent, there is at least one coefficient from $a_m, \dots, a_n$ is non-zero, say $a_m \neq 0$. Then by 3.B.1,

$$w_1, \dots, w_{m-1}, w_m, v_{m+1}, \dots, v_n$$

is a basis of V.

<u>3.B.3 Theorem</u> Let V be a K-vector space which has a finite basis $v_1, \dots, v_n$. Then every basis of V is finite and has exactly n elements.

<u>Proof</u> By 3.B.2 every linearly independent family in V has at most n elements. So every basis of V has at most n elements.

Further, there is no basis ~~with~~ of cardinality less than n, since otherwise by 3.B.2 we can extend this to a basis of cardinality n.

3.B.4 Definition Let V be a K-vector space.

(1) If V has a finite basis, then V is called *finite dimensional* and the uniquely determined natural number n = the number of elements in a basis of V is called the *dimension* of V and is denoted by

$$\mathrm{Dim}_K V \text{ or simply by } \mathrm{Dim}\, V.$$

If $\mathrm{Dim}_K V = n$, then V is called *n-dimensional*.

(2) If V does not have finite basis, then V is called *infinite dimensional*. In this case we write $\mathrm{Dim}_K V = \mathrm{Dim}\, V = \infty$.

If V has a countable infinite basis, then we also write

$$\mathrm{Dim}_K V = \mathrm{Dim}\, V = \aleph_0 = |\mathbb{N}|$$

and if V ~~has~~ does not have countable basis then $\mathrm{Dim}_K V > \aleph_0$

3.B.5 Examples (1) Let I be a set and let K be a field. By Example 3.A.5, the K-vector space $K^{(I)}$ is finite dimensional if and only if I is finite. In this case $K^{(I)} = K^I$ and $\mathrm{Dim}_K K^{(I)} = |I| =$ the number of elements in I.

In particular, $\mathrm{Dim}_K K^n = n$ for all natural number. If I is countable infinite, then $\mathrm{Dim}_K K^{(I)} = |I| = \aleph_0$. If I is uncountable, then $\mathrm{Dim}_K K^{(I)} > \aleph_0$ (See Exercise 20b) of 3.A)

(2) Let K be a field. The space $K^{\mathbb{N}}$ of sequences in K does not have countable generating system, this follows from the following: Let I be countable infinite, then $\mathrm{Dim}_K K^{(I)} = \aleph_0$ and if I is infinite, then $\mathrm{Dim}_K K^I > \aleph_0$ (See Exercise 3A 20b))

(3) Let V be a $\mathbb{C}$-vector space. By Example 3.A.12, V is finite dimensional if and only if V is finite dimensional as $\mathbb{R}$-vector space. In this case $\mathrm{Dim}_{\mathbb{R}} V = 2 \cdot \mathrm{Dim}_{\mathbb{C}} V$. More generally, by 3.A, Exercise 28: If $K \subseteq L$ is a field extension with $\mathrm{Dim}_K L = m < \infty$, then an L-vector space V is finite dimensional if and only if V is finite dimensional as K-vector space. In this case $\mathrm{Dim}_K V = m \cdot \mathrm{Dim}_L V$. The dimension $\mathrm{Dim}_K L$ is called the degree of the field extension $K \subseteq L$ and is usually denoted by $[L:K]$. For example $[\mathbb{C}:\mathbb{R}] = 2$ and $[\mathbb{R}:\mathbb{Q}]\ (=\aleph) > \aleph_0$.

(4) Let K be an infinite field. Then the space $K[t]_n$ of the polynomial functions of degree $< n$ has the dimension n for every $n \in \mathbb{N}$, see Example 3.A.7.

(5) Let $I \subseteq \mathbb{R}$ be an interval with more than one point. For all $\alpha \in \mathbb{N} \cup \{\infty, \omega\}$, $\operatorname{Dim}_{\mathbb{K}} C^{\alpha}_{\mathbb{K}}(I) > \aleph_0$, see 3.A. Exercise 21.

(6) Let V and W be K-vector spaces with bases v_i, $i \in I$ and w_j, $j \in J$, respectively. Then clearly $(v_i, 0)$, $i \in I$ and $(0, w_j)$, $j \in J$ together form a basis of $V \oplus W = V \times W$. In particular, $V \oplus W$ is finite dimensional if and only if both V and W are finite dimensional. In this case $\operatorname{Dim}_K V \oplus W = \operatorname{Dim}_K V + \operatorname{Dim}_K W$

Analogously, one can show that for arbitrary finite dimensional K-vector spaces $V_1, \dots, V_n$ the direct sum $V_1 \oplus \cdots \oplus V_n$ is also finite dimensional with $\operatorname{Dim}_K (V_1 \oplus \cdots \oplus V_n) = \operatorname{Dim}_K V_1 + \cdots + \operatorname{Dim}_K V_n$

3.B.6 Example Let $P \in \mathbb{K}[x]$ be a monic polynomial of degree $n \in \mathbb{N}$ and let $I \subseteq \mathbb{R}$ be an interval with more than one point. Then the $\mathbb{K}$-vector space V of the solutions of the homogeneous linear differential equations

$$P(D)y = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = 0$$

has dimension n. Every basis of this vector space is called a <u>fundamental system</u> of solutions of the differential equation. If $P(x) = (X-\lambda_1)^{n_1} \cdots (X-\lambda_r)^{n_r}$ with pairwise distinct $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$, $n_1, \ldots, n_r \in \mathbb{N}^+$. Then the quasi-polynomials

$$e^{\lambda_1 t}, \ldots, t^{n_1-1}e^{\lambda_1 t}, \ldots, e^{\lambda_r t}, \ldots, t^{n_r-1}e^{\lambda_r t}$$

is a $\mathbb{K}$-basis, i.e. a fundamental system of solutions of $P(D)y = 0$, cf. Vol 1, 19.D.6. For further assertions see Vol 1, 19.D.8 and 19.D, Exercises 4 and 5 and Section 11.E of Vol 2

If the dimension n of a finite dimensional vector space V is known, then to check that a system of n elements $v_1, \ldots, v_n$ in V is a basis of V, it is enough only to check only one of the both basis-properties, namely, either $v_1, \ldots, v_n$ are linearly independent or $v_1, \ldots, v_n$ is a generating system for V:

3.B.7 Theorem Let V be an n-dimensional K-vector space, $n \in \mathbb{N}$. For a system $v_1, \ldots, v_n$ of n elements in V, the following statements are equivalent:

(i) $v_1, \ldots, v_n$ is a basis of V.

(ii) $v_1, \ldots, v_n$ is a generating system for V.

(iii) $v_1, \ldots, v_n$ is linearly independent.

Proof The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are trivial.

(ii) $\Rightarrow$ (i): The generating system $v_1, \dots, v_n$ contains a basis by Corollary to 3.A.15 and by 3.B.3 every basis of V has exactly n elements. So $v_1, \dots, v_n$ is a basis of V.

(iii) $\Rightarrow$ (i): The linearly independent system $v_1, \dots, v_n$ can be extended to a basis of V by 3.B.2, but by 3.B.3 every basis of V has exactly n elements. So $v_1, \dots, v_n$ is a basis of V.

3.B.8 Example (Partial fractions)

We once again prove here Theorem 11.B.2 of Volume 1 on the partial fractions: Let K be an infinite field, $\alpha_1, \dots, \alpha_r \in K$ distinct elements in K, $n_1, \dots, n_r \in \mathbb{N}^+$ +ve natural numbers with $n := n_1 + \cdots + n_r$ and G be the polynomial function

$$G := (t-\alpha_1)^{n_1} \cdots (t-\alpha_n)^{n}.$$

Then the rational functions

$$\frac{1}{t-\alpha_1}, \ldots, \frac{1}{(t-\alpha_1)^{n_1}}, \ldots, \frac{1}{t-\alpha_r}, \ldots, \frac{1}{(t-\alpha_r)^{n_r}}$$

defined on $K \setminus \{\alpha_1, \ldots, \alpha_r\}$ and is a basis of the vector space of all rational functions of the form F/G, where F is a polynomial function of degree $< n$.

Proof Since the vector space $V = \{F/G \mid F \in K[t]_n\}$ has dimension n, by 3.B.7 it is enough to prove that the above given rational functions are linearly independent. For this, suppose that

$$R := \frac{\alpha_{11}}{(t-\alpha_1)} + \cdots + \frac{\alpha_{1n_1}}{(t-\alpha_1)^{n_1}} + \cdots + \frac{\alpha_{r1}}{(t-\alpha_r)} + \cdots + \frac{\alpha_{rn_r}}{(t-\alpha_r)^{n_r}}$$

$= 0$ with $\alpha_{11}, \ldots, \alpha_{1n_1}, \ldots, \alpha_{r1}, \ldots, \alpha_{rn_r} \in K$ and $\alpha_{ij} \neq 0$, $\alpha_{i,j+1} = \cdots = \alpha_{in_i} = 0$ for some index pair i, j. Now, multiplying the equation $R=0$ by $G/(t-\alpha_i)^{n_i}$

We get $(t-\alpha_i)H + \alpha_{ij}G_i = 0$

with a polynomial function H and $G_i := \prod_{k\neq i}(t-\alpha_k)^{n_k}$. In this polynomial equation, put $t = \alpha_i$ to conclude that $\alpha_{ij}G_i(\alpha_i) \neq 0$ in K, but $G_i(\alpha_i) \neq 0$. Therefore $\alpha_{ij} = 0$ which is a contradiction to our choice that $\alpha_{ij} \neq 0$.

From 3.B.2 it follows that:

3.B.9 Theorem Let V be a K-vector space and $n \in \mathbb{N}$. Then

(1) V is finite dimensional of $\mathrm{Dim}_K V \leq n$ if and only if every $n+1$ vectors in V are linearly dependent.

(2) V is finite dimensional of $\mathrm{Dim}_K V = n$ if and only if every $n+1$ vectors in V are linearly dependent and there are n linearly independent vectors in V.

(3) V is finite dimensional of $\mathrm{Dim}_K V \leq n$ if and only if V has a generating system consisting n vectors.

(4) V is finite dimensional of $\mathrm{Dim}_K V = n$ if and only if V has a generating system consisting n vectors and (for $n>0$) every generating system of V has at least n elements.

In the following theorem we study the dimension of subspace in a finite dimensional vector space:

3.B.10 Theorem Let V be a finite dimensional K-vector space and let $W \subseteq V$ be a subspace of V. Then:

(1) W is also finite dimensional and
$$\mathrm{Dim}_K W \leqslant \mathrm{Dim}_K V.$$

(2) Every basis of W can be extended

(3) $W = V$ if and only if $\mathrm{Dim}_K W = \mathrm{Dim}_K V$.

Proof Let $n := \mathrm{Dim}_K V$. Then every $n+1$ vectors in V are linearly dependent and hence every $(n+1)$ vectors in W are also linearly independent and hence $\mathrm{Dim}_K W \leq n$ by 3.B.9 (1). This proves (1). (2) and (3) directly follow from 3.B.2 (Steinitz's exchange theorem)

The proof of the following theorem is a typical example of how one can work with bases.

3.B.11 Dimension Formula. Let U and W be finite dimensional subspace of a K-vector space V. Then $U \cap W$ and $U + W$ are also finite dimensional and

$$\mathrm{Dim}_K(U+W)+\mathrm{Dim}_K(U\cap V)$$
$$= \mathrm{Dim}_K U + \mathrm{Dim}_K V.$$

Proof $U\cap W$ is a subspace of U (or W) and hence finite dimensional. Let $x_1, \dots, x_r$ be a basis of $U\cap W$. Extend this to bases

$x_1, \dots, x_r, u_1, \dots, u_s$ resp. $x_1, \dots, x_r, w_1, \dots, w_t$

of U resp. W. It is enough to show that

$$x_1, \dots, x_r, u_1, \dots, u_s, w_1, \dots, w_t$$

is a basis of $U+W$.

First, we shall show that they are linearly independent: from

$$\underbrace{a_1x_1+\cdots+a_rx_r+b_1u_1+\cdots+b_su_s+c_1w_1+\cdots+c_tw_t}_{=0}$$

it follows that

$$x := a_1x_1+\cdots+a_rx_r+b_1u_1+\cdots+b_su_s = -c_1w_1-\cdots-c_tw_t$$
$$\in U\cap W$$

and hence x is a linear combination of $x_1, \dots, x_r$ only and hence $b_1 = \cdots = b_s = 0$.

Now, from $a_1x_1+\cdots+a_rx_r+c_1w_1+\cdots+c_tw_t=0$, and linear independence of $x_1,\dots,x_r,w_1,\dots,w_t$ it follows that $a_1=\cdots=a_r=c_1=\cdots=c_t=0$.

The vectors $x_1,\dots,x_r,u_1,\dots,u_s,w_1,\dots,w_t$ generate $U+W$. The subspace gen. by $x_1,\dots,x_r,u_1,\dots,u_s,w_1,\dots,w_t$ is clearly contained in $U+W$. Moreover, it contains the subspaces $U=Kx_1+\cdots+Kx_r+Ku_1+\cdots+Ku_s$ and $W=Kx_1+\cdots+Kx_r+Kw_1+\cdots+Kw_t$ and hence also contains $U+W$.

3.B.12 Examples

(1) Let V be 1-dimensional K-vector space. Then subspaces of V can have dimensions 0 or 1 only. Therefore 0 and V are the only subspaces of V.

(2) Let V be a 2-dimensional K-vector space. Other than the trivial subspaces 0 and V of dimensions 0 and 2, V has only 1-dimensional subspaces Kx, where x is a vector in V, $x\neq 0$. Analogously,

in a 3-dimensional K-vector space V, other than 0, V and 1-dimensional subspaces Kx, $x \in V$, $x \neq 0$, there are the 2-dimensional subspaces $Kx + Ky$, where $x, y \in V$ are linearly independent vectors in V, i.e. such vectors are not multiples of the other.

(3) (Flags) Let $v_1, \ldots, v_n$ be a basis of the n-dimensional K-vector space V. Then the subspaces $V_i := Kv_1 + \cdots + Kv_i$, $i = 0, \ldots, n$, form a chain of subspaces:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

with $\mathrm{Dim}_K V_i = i$, $i = 0, \ldots, n$. This chain of subspaces in V has maximal length. For, if $U_0 \subset U_1 \subset \cdots \subset U_m \subseteq V$ is a proper ascending chain of subspaces in V, then by 3.B.10 (3) $\mathrm{Dim}_K U_{i-1} < \mathrm{Dim}_K U_i$ and hence $\mathrm{Dim}_K U_i \geq i$ for all $i = 0, \ldots, m$ and so $m \leq \mathrm{Dim}_K U_m \leq \mathrm{Dim}_K V = n$.

A maximal proper ascending chain

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = V$$

of subspaces in V, necessarily have $\mathrm{Dim}_K W_i = i$, for all $i = 0, \ldots, n$.
Such a chain is called a Flag in V.
For such a chain if we choose $v_i \in W_i$, $v_i \notin W_{i-1}$, $i = 1, \ldots, n$, then $W_i = \sum_{j=1}^{i} K v_j$, $i = 0, \ldots, n$ and $v_1, \ldots, v_n$ is a basis of V.

3.B.13 Examples (1) We use the notation as in 3.B.11. From the dimension formula it follows that

$$\mathrm{Dim}_K (U + W) \leq \mathrm{Dim}_K U + \mathrm{Dim}_K W,$$

moreover, the equality holds if and only if $\mathrm{Dim}_K U \cap W = 0$, i.e. $U \cap W = 0$; in this we say that $U + W$ is a direct sum of U and W. We shall this important concept in section § 5.F extensively.

If V is also finite dimensional with $\mathrm{Dim}_K V = n$, then $\mathrm{Dim}_K U+W \leq \mathrm{Dim}_K V = n$ and from dimension formula we get the inequality:

$$\mathrm{Dim}_K(U \cap W) \geq \mathrm{Dim}_K U + \mathrm{Dim}_K W - n.$$

In particular, If the sum $\mathrm{Dim}_K U + \mathrm{Dim}_K W$ of the dimensions of U and W is bigger than the dimension of V, then their intersection $U \cap W$ is not the zero-space.

(2) (Codimension - hyperplanes) Let V be a K-vector space of the finite dimension $n \in \mathbb{N}$. For a subspace U of V, $n - \mathrm{Dim}_K U$ is called the codimension of U in V and is denoted by

$$\mathrm{Codim}_V U = \mathrm{Codim}(U, V)$$

From the dimension formula 3.B.11 we get the codimension formula:

$$\mathrm{Codim}_V(U+W) + \mathrm{Codim}_V(U \cap W) = \mathrm{Codim}_V U + \mathrm{Codim}_V W$$

for arbitrary subspaces $U, W \subseteq V$.
In particular,

$$\operatorname{Codim}_V (U \cap W) \leq \operatorname{Codim}_V U + \operatorname{Codim}_V W$$

Moreover, the equality holds if and only if $\operatorname{Codim}_V (U+W) = 0$, i.e. if and only if $U + W = V$. In this case we say that U and W are transversal in V.

A subspace of codimension 1 in V, i.e. dimension $\operatorname{Dim}_K V - 1 = n-1$ is called a hyperplane in V. If H is a hyperplane in V, then other than H and V there is no subspace U in V with $H \subseteq U \subseteq V$, since each of such a subspace has dimension $n-1$ or n.
Let H_1 and H_2 be two distinct hyperplanes in V. Then $H_1 + H_2$ is a subspace of V and $H_1 \subsetneq H_1 + H_2$, ($H_2 \subsetneq H_1 + H_2$) and hence $H_1 + H_2 = V$. Now, from the codimension formula, it follows that

$$\operatorname{Codim}_V H_1 \cap H_2 = \operatorname{Codim}_V H_1 + \operatorname{Codim}_V H_2 = 2.$$

In words: Two distinct hyperplanes intersects transversally in a subspace of codimension 2.