5.B Linear Maps

Let V and W be vector spaces over a field K. Homomorphisms of V into W not only respect the additions, but also the scalar-multiplications.

5.B.1 Definition Let V and W be vector spaces over a field K. A map $f: V \to W$ is called K-linear or an K-homomorphism if the following two conditions are satisfied:

(1) f is additive, i.e. a homomorphism of the additive group $(V, +)$ of V into the additive group $(W, +)$ of W. Therefore, for all $x, y \in V$, we have

$$f(x+y) = f(x) + f(y)$$

(2) For all $a \in K$ and all $x \in V$,

$$f(ax) = af(x)$$

The conditions (1) and (2) for a homomorphism $f: V \longrightarrow W$ can be combined into the following condition together For all $a, b \in K$ and all $x, y \in V$, we have

$$f(ax+by) = af(x) + bf(y).$$

From this, by induction, it follows that

$$f(a_1x_1 + \cdots + a_nx_n) = a_1f(x_1) + \cdots + a_nf(x_n)$$

for all $a_1, \ldots, a_n \in K$ and all $x_1, \ldots, x_n \in V$.

More generally,

$$f\Big(\sum_{i\in I} a_ix_i\Big) = \sum_{i\in I} a_if(x_i),$$

where x_i, $i \in I$, is a family of vectors in V and $(a_i) \in K^{(I)}$ is a coefficient system.

As in the case of groups (see Definition 5.A.1), we can also define[1] isomorphisms, Endomorphisms and Automorphisms for vector spaces. An endomorphism $f: V \longrightarrow V$ of the K-vector space V is also called an K-linear operator on V. A K-homomorphism $f: V \longrightarrow K$ of V into the field of scalars K is called a K-linear form on V

[1] Bijective homomorphisms $f: V \longrightarrow W$ are called isomorphisms. Homomorphisms of V into V are called endomorphisms; bijective endomorphisms are isomorphisms of V onto V and are called automorphisms of V.

The set of all homomorphisms of a K-vector space V in a K-vector space W is denoted by

$$\mathrm{Hom}_K(V, W);$$

the set of all endomorphisms of V is denoted by

$$\mathrm{End}_K V$$

and the set of all linear forms on V is denoted by

$$V^*.$$

Therefore we have:

$$\mathrm{End}_K V = \mathrm{Hom}_K(V, V) \text{ and } V^* = \mathrm{Hom}_K(V, K)$$

Two endomorphisms f, g in $\mathrm{End}_K V$ are said to be <u>commuting</u> if $f \circ g = g \circ f$.
In the case, there is no confusion about the field of scalars K, for the sets $\mathrm{Hom}_K(V, W)$ and $\mathrm{End}_K V$, one can simply write $\mathrm{Hom}(V, W)$ and $\mathrm{End}\, V$ respectively.
Since vector space homomorphisms are in particular, group homomorphisms,

all results in the earlier subsection 5.A also hold. However, note that the group operation in vector spaces is written additively. In particular, $f(0)=0$ for every homomorphism f of vector spaces.

Further, (see 5.A.3): The composition of K-vector space homomorphisms is also a K-vector space homomorphism and the inverse of a K-vector space isomorphism is again a K-vector space isomorphism.

The isomorphism $V \cong W$ of K-vector spaces is again is an equivalence relation on the (class) set of K-vector spaces. The automorphisms of a K-vector space V form a subgroup

$$\mathrm{Aut}_K V = \mathrm{Aut}\, V$$

of the permutation group $\mathfrak{S}(V)$ of V. This automorphism group is also denoted by

$$GL_K(V) = GL(V),$$

in the case $V := K^{(I)}$ by $GL_I(K)$ and for

$V := K^n$ by $GL_n(K)$. The group $GL_K(V)$ is called the (general) Lineargroup of V

The analogs of 5.A.4, we formulate explicitly:

5.B.2 Theorem Let $f: V \longrightarrow W$ be a homomorphism of K-vector spaces.

(1) If V' is a K-subspace of V, then the image $f(V')$ of V' is a K-subspace of W. In particular, the image $f(V)$ of V is a K-subspace of W.

(2) If W' is a K-subspace of W, then the inverse image $f^{-1}(W')$ of W' is a K-subspace of V. In particular, $\operatorname{Ker} f = f^{-1}(0)$ is a K-subspace of V.

Proof (1) For $f(x), f(x') \in f(V')$, $x, x' \in V'$, clearly $f(x) + f(x') = f(x+x') \in f(V')$, since $x+x' \in V'$. Further, for all $a \in K$, $af(x) = f(ax) \in f(V')$, since $ax \in V'$.

(2) For $x, x' \in f^{-1}(W')$, $x, x' \in V$, we have

$f(x), f(x') \in W'$ and hence
$f(x+x') = f(x)+f(x') \in W'$, i.e. $x+x' \in \overset{-1}{f}(W')$.
Further, for all $a \in K$, $f(ax) = af(x) \in W'$, i.e. $ax \in \overset{-1}{f}(W')$. ∎

By 5.A.5 a K-vector space homomorphism $f: V \longrightarrow W$ is injective if and only if $\operatorname{Ker} f = 0$. More generally, (by 5.A.6) for an arbitrary K-vector space homomorphism $f: V \longrightarrow W$, we have the equality

$$\overset{-1}{f}(y) = x + \operatorname{Ker} f$$

in the case $x \in V$ is an element of the fibre $\overset{-1}{f}(y)$, $y \in W$.
Therefore all elements of the fibre $\overset{-1}{f}(y)$ can be obtained by adding a special element x of the fibre $\overset{-1}{f}(y)$ to the elements of the kernel $\operatorname{Ker} f$ of f.
In other words: The fibres of f are affine subspaces of V which are parallel to $\operatorname{Ker} f$.

5.B.3 Examples (1) Let D be any set and let K be a field. For a fixed $t_0 \in D$, the substitution map

$$\varphi_{t_0} : K^D \longrightarrow K, \quad x \longmapsto x(t_0)$$

is a K-linear form on the K-vector space K^D of all K-valued functions on I

The kernel is all those functions $x : D \to K$ for which t_0 is a zero, i.e.

$$\operatorname{Ker} \varphi_{t_0} = \{ x \in K^D \mid x(t_0) = 0 \}.$$

(2) Let $I \subseteq \mathbb{R}$ be an interval with more than one point. The differentiation define a $\mathbb{K}$-linear map

$$D : C^1_{\mathbb{K}}(I) \longrightarrow C^0_{\mathbb{K}}(I), \quad D(x) = x'.$$

Moreover, D is surjective, since every continuous function on I has a primitive

The kernel of D is the 1-dimensional subspace consisting of the constant functions on I.

Let $t_0 \in I$ be fixed. Then the integration define a $\mathbb{K}$-linear map

$$S : C^0_{\mathbb{K}}(I) \longrightarrow C^1_{\mathbb{K}}(I), \quad Sx := \left(t \mapsto \int_{t_0}^{t} x(\tau)\,d\tau\right)$$

Moreover, the composition with D is

$$D \circ S = id_{C^0_{\mathbb{K}}(I)} \text{ and } S \circ D = (x \mapsto x - x(t_0)).$$

The map S is injective and hence $\operatorname{Ker} S = 0$. The image of S is the functions in $C^1_{\mathbb{K}}(I)$ which vanish at t_0.

(3) On the $\mathbb{K}$-vector space $C^0_{\mathbb{K}}([a,b])$ of all continuous $\mathbb{K}$-valued functions on the interval $[a,b] \subseteq \mathbb{R}$, the definite integral $x \longmapsto \int_a^b x(t)\,dt$ is a $\mathbb{K}$-linear form.

<u>**5.B.4 Example**</u> (<u>Linear System of equations</u>) Let K be a field. We consider a K-linear map $f : K^n \longrightarrow K^m$. For every vector

$$x = (x_1, \dots, x_n) = \sum_{j=1}^{n} x_j e_j$$

where $e_1, \dots, e_n$ is the standard basis of K^n, We have

$$f(x) = f\left(\sum_{j=1}^{n} x_j \cdot e_j\right) = \sum_{j=1}^{n} x_j \cdot f(e_j).$$

Therefore f is uniquely determined by the images $f(e_1), \dots, f(e_n)$. Suppose that these vectors are

$$f(e_j) = (a_{1j}, \dots, a_{mj}), \; j = 1, \dots, n.$$

Then the components $y_1, \dots, y_m$ of $y = f(x)$ are represented by the system of m equations in $x_1, \dots, x_n$, i.e

$$\begin{aligned} y_1 &= a_{11} x_1 + \cdots + a_{1n} x_n \\ y_2 &= a_{21} x_1 + \cdots + a_{2n} x_n \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \\ y_m &= a_{m1} x_1 + \cdots + a_{mn} x_n . \end{aligned}$$

Conversely, for arbitrary $a_{ij} \in K$, $i = 1, \dots, m$, $j = 1, \dots, n$, the map defined by $x \longmapsto y$ from $K^n \longrightarrow K^m$ is K-linear.

The kernel of f is the solution space L_0 of the homogeneous system of equations:

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + \cdots + a_{2n}x_n = 0$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = 0$$

More generally, the fibre $\overset{-1}{f}(b)$ for $b = (b_1, \ldots, b_m) \in K^m$ is the solution-space of the system of linear equations:

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots + a_{2n}x_n = b_1$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m.$$

The assertion 2.A.1 is a special case of 5.A.6. The system of linear equations has a solution if and only if $b \in \operatorname{Im} f$. Further, $\operatorname{Im} f$ is a subspace of K^n generated by $a_1 = f(e_1), \ldots, a_n = f(e_n)$.

By 3.B.10 (3) the given system of linear equations has at least one solution if and only if
$$\operatorname{Dim}_K (Ka_1 + \cdots + Ka_n) = \operatorname{Dim}_K (Ka_1 + \cdots + Ka_n + Kb).$$

This example shows that: the study of finitely many linear equations in finitely many unknowns can be reduced to the study of linear maps between finite dimensional vector space. We shall frequently use this connection.

5.B.5 Example Let v_i, $i \in I$, be a basis of the K-vector space V. Then the coordinate functions v_i^*, $i \in I$, are linear forms on V.

5.B.6 Example Let U_i, $i \in I$, be a family of subspaces of the K-vector space V. The map

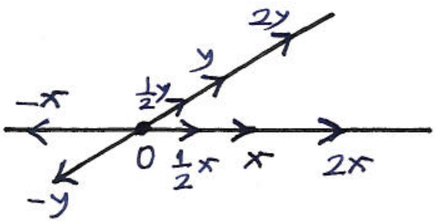
$$\bigoplus_{i \in I} U_i \longrightarrow V, \quad (u_i)_{i \in I} \longmapsto \sum_{i \in I} u_i$$

is a K-linear map from the direct sum $\bigoplus_{i \in I} U_i$ of U_i, $i \in I$, into the space V; its image is precisely the sum $\sum_{i \in I} U_i$ of U_i, $i \in I$. More generally, let $f_i : V_i \longrightarrow V$, $i \in I$, be a family of K-vector space homomorphisms. Then the map

$$f : \bigoplus V_i \longrightarrow V, \quad (x_i)_{i\in I} \longmapsto \sum_{i\in I} f_i(x_i)$$

is a K-linear map from the direct sum $\bigoplus_{i\in I} V_i$ into the vector space V; its image is the sum $\sum_{i\in I} \operatorname{Im} f_i$ of the family $\operatorname{Im} f_i$, $i\in I$, of subspaces of V. The map f is called the sum of the family f_i, $i \in I$.

5.B.7 Example (Homotheties) Let V be a K-vector space. For a fixed $a\in K$, the map $\vartheta_a : V \longrightarrow V$, $x \longmapsto ax$, is clearly an K-endomorphism of V. This map ϑ_a is called the homothety or stre___ of V with the stretching factor a. It is $\vartheta_a = a\cdot \mathrm{id}_V$.



5.B.8 Example Let Ω be a discrete probability space[1]. The random variables[1]

$X : \Omega \longrightarrow \mathbb{K}$ for which the expectation value[1] $E(X)$ exists, form a subspace $\mathcal{L}^1_{\mathbb{K}}(\Omega)$ of $\mathbb{K}^{\Omega}$. Moreover, the map

$\mathcal{L}^1_{\mathbb{K}}(\Omega) \longrightarrow \mathbb{K}$, $X \longmapsto E(X)$, is a $\mathbb{K}$-linear form on $\mathcal{L}^1_{\mathbb{K}}(\Omega)$.

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[1] A discrete probability space (Ω, P) is a set Ω with a (summable) function $P : \Omega \longrightarrow [0,1]$ such that $\sum_{\omega \in \Omega} P(\omega) = 1$.

The elements of Ω are called elementary samples, Ω is called the sample space and P is called the probability function or the (probability)-distribution. For $\omega \in \Omega$, $P(\omega)$ is called the probability of ω.

A map $X : \Omega \longrightarrow \mathbb{K}$ is called a random-variable on Ω with values in $\mathbb{K}$; it is therefore an element of the $\mathbb{K}$-vector space $\mathbb{K}^{\Omega}$.

For random-variables the expectation- or mean-value is an important invariant. However it may not always exist!

For a $\mathbb{K}$-valued random variable $X \in \mathbb{K}^{\Omega}$,

5.B.9 Remark Let E be an affine space over the K-vector space V and let W be another K-vector space. A map $f: V \to W$ will be interpreted as follows:
For all $P, Q \in E$, $f(\overrightarrow{PQ})$ is the W-valued change F by transferring the point P to the point Q with the help of the translation by $\overrightarrow{PQ}$. Then, let $P = P_0, P_1, \ldots, P_r = Q$ be arbitrary points in E. Under the transition of $P = P_0$ to $Q = P_r$, the intermediate points $P_1, \ldots, P_{r-1}$ with the help of translations by $\overrightarrow{P_0P_1}, \overrightarrow{P_1P_2}, \ldots, \overrightarrow{P_{r-1}P_r}$, altogether change to $f(\overrightarrow{P_0P_1}) + f(\overrightarrow{P_1P_2}) + \cdots + f(\overrightarrow{P_{r-1}P_r})$.

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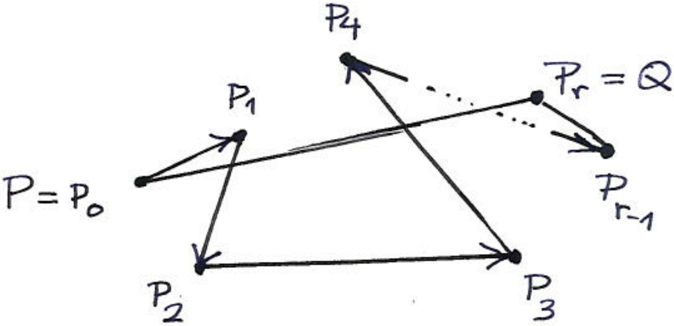
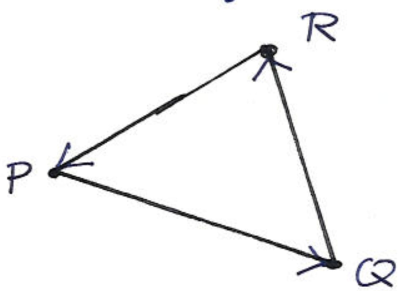
if the family $X(\omega)P(\omega)$, $\omega \in \Omega$ is summable, then X is called a random-variable with expectation-value and the sum $E(X) := \sum_{\omega \in \Omega} X(\omega)P(\omega)$ is called the expectation-value of X.
For example, let (Ω, P) be the discrete probability space with $\Omega = \mathbb{N}^*$ and $P(n) = 1/2^n$. Then for the random-variable $X: \Omega \to \mathbb{R}$, with $X(n) := \sqrt{2^n}$. Then the expectation-value $E(X)$ exists, but $E(X^2)$ does not exist.



Since $\overrightarrow{P_0P_1} + \overrightarrow{P_1P_2} + \cdots + \overrightarrow{P_{r-1}P_r} = \overrightarrow{P_0P_r} = \overrightarrow{PQ}$, it follows that: The change F by transferring the point P to Q is independent of the choice of path from P to Q if and only if f is additive.

The independence is already ensured if for every "triangular-path" P, Q, R, P, the total change is 0.



If f is additive, then the change F by transferring the point P to Q is independent of the path from P to Q, is equal to

$$U(Q) - U(P),$$

where $U : E \longrightarrow W$ is defined by $P \longmapsto f(\overrightarrow{OP}) + w$, with an arbitrary but fixed origin $O \in E$ and an arbitrary but fixed vector $w = U(O) \in W$. These maps U differ only by an additive constants and are so-called <u>affine maps</u> $E \longrightarrow W$ with linear part $f : V \longrightarrow W$, see subsection 7.A, and are also called the <u>primitive functions</u> of f.

This aspect of linear maps, changes by transitions, arise in more general form in Analysis as the concept of differential forms.

Finally, we remark that in the case $K = \mathbb{R}$ and finite dimensional $\mathbb{R}$-vector spaces V and W, the $\mathbb{R}$-linearity of a map $f : V \longrightarrow W$ already follows from the additivity of f, if, in addition, we assume a natural continuity condition on f.

5.C. Space of Linear Maps

Let V and W be arbitrary vector spaces over a field K. Then the set $\mathrm{Hom}_K(V,W)$ of all K-homomorphisms from V into W is a subset of the K-vector space W^V of all maps from V into W. Analog of 5.A.12 is:

5.C.1 Theorem Let V and W be K-vector spaces. Then $\mathrm{Hom}_K(V,W)$ is a K-subspace of the K-vector space W^V of all maps from V into W, i.e. we have:

(1) The zero-map 0 belongs to $\mathrm{Hom}_K(V,W)$.

(2) If $f, g \in \mathrm{Hom}_K(V,W)$, then $f+g \in \mathrm{Hom}_K(V,W)$.

(3) If $f \in \mathrm{Hom}_K(V,W)$, then $af \in \mathrm{Hom}_K(V,W)$ for every $a \in K$.

Proof (1) is trivial; the zero-map 0 is clearly K-linear.

(2) For all $a, b \in K$ and all $x, y \in V$, we have:

$(f+g)(ax+by) = f(ax+by) + g(ax+by) =$

$$bf(x) + bf(y) + ag(x) + bg(y)$$
$$= b(f+g)(x) + c(f+g)(y),$$

i.e. $f+g$ is also K-linear.

(3) For all $b, c \in K$ and all $x, y \in V$, we have: $(af)(bx+cy) = af(bx+cy)$
$$= a(bf(x) + cf(y)) \overset{*}{=} b(af)(x) + c(af)(y),$$
i.e. af is also K-linear. (Note here that for $*$ we have used the commutativity of multiplication in K.)

The similar simple proofs of the following computation-rules are left to the reader (see proof of 5.A.13):

5.C.2 Theorem Let U, V, W be vector spaces over a field K. For arbitrary homomorphisms $f, f' : U \longrightarrow V$ and $g, g' : V \longrightarrow W$ and all $a, b \in K$, we have:

(1) $f \circ (g+g') = f \circ g + f \circ g'$ and
$$(f+f') \circ g = f \circ g + f' \circ g.$$

(2) $(af) \circ (bg) = (ab)(f \circ g)$.

From 5.C.2 it follows that: for a K-vector space V, the set $\mathrm{End}_K V$ of all

K-endomorphisms of V is not only a K-vector space, but with the (vector space) addition and the composition as multiplication is also a ring with the identity id_V as the unit-element. Moreover, the ring multiplication and the scalar-multiplication on $\mathrm{End}_K V$ are compatible, see 5.C.2 (2). Such a situation occurs in many places. Therefore, we make the following general definition:

<u>5.C.3 Definition</u> Let K be a field and let A be a K-vector space, moreover, a multiplication $A \times A \longrightarrow A$ is defined. Then A is called a <u>K-algebra</u> if the following conditions are satisfied:

(1) A is with the vector space addition and the given multiplication is a ring.

(2) For all $a, b \in K$ and all $x, y \in A$, we have $\quad (ax)(by) = (ab)(xy)$

If A and B are two K-algebras, then a map $\varphi : A \longrightarrow B$ is called a K-algebra homomorphism if

(1) φ is a K-vector space homomorphism.

(2) φ is compatible with the multiplications on A and B, i.e.

$$\varphi(xy) = \varphi(x)\,\varphi(y) \text{ for all } x, y \in A,$$

and moreover, $\varphi(1_A) = 1_B$.

In particular, a K-algebra homomorphism is also a ring homomorphism.

From 5.C.1 and 5.C.2 it follows that:

5.C.4 Theorem Let K be a field and let V be a K-vector space. Then $\mathrm{End}_K V$ is a K-algebra. Moreover, the unit-group $(\mathrm{End}_K V)^\times$ of $\mathrm{End}_K V$ is the automorphism group $\mathrm{Aut}_K V$ of V.

5.C.5 Example (Function-Algebras)

An important class of (commutative) algebras is formed by the function-algebras. For an arbitrary field K and an arbitrary set D, the set K^D of all

K-valued functions on D is a commutative K-algebra in a natural way and the substitution maps (see Example 5.B.3(1) are K-algebra homomorphisms. All examples (of subspaces) in 1.C.10 (4), (5), (6) and in 1.C, Exercise 12, are subalgebras of algebras of the type K^D. -- A subset A' of an K-algebra A is called a K-subalgebra of A if A' is a K-subspace as well as a subring of A. However, the subspace $K[t]_n$ of $K[t]$ given in 1.C.10(6) is not a subalgebra of $K[t]$.

5.C.6 Example The field of complex numbers $\mathbb{C}$ is a 2-dimensional $\mathbb{R}$-algebra with the $\mathbb{R}$-basis $1, i$. It has only the trivial $\mathbb{R}$-subalgebras $\mathbb{R} = \mathbb{R} \cdot 1_{\mathbb{C}}$ and $\mathbb{C}$.

5.C.7 Example ($\mathbb{C}$-anti-linear maps) Let V and W be two complex vector-spaces (over $\mathbb{C}$), we consider them also as $\mathbb{R}$-vector spaces in a natural way, see Example 1.B.5. A map $h: V \longrightarrow W$ is

called $\mathbb{C}$-*antilinear* (or $\mathbb{C}$-*semi-linear*) if h is additive and if
$$h(zx) = \bar{z}h(x)$$
for all $z \in \mathbb{C}$ and all $x \in V$. The set of all $\mathbb{C}$-anti-linear maps from V into W is denoted by $\mathrm{Hom}_{\overline{\mathbb{C}}}(V,W)$.

A map $h: V \longrightarrow W$ is clearly $\mathbb{C}$-linear (resp. $\mathbb{C}$-anti-linear) if and only if h is $\mathbb{R}$-linear and for all $x \in V$, we have $h(ix) = ih(x)$ (resp. $h(ix) = -ih(x)$).

Like the $\mathbb{C}$-linear maps, the $\mathbb{C}$-anti-linear maps from V into W, also form a $\mathbb{C}$-subspace of W^V.

Now, let $f: V \longrightarrow W$ be an arbitrary $\mathbb{R}$-linear map. Then
$$f = f_{\mathbb{C}} + f_{\overline{\mathbb{C}}}$$
with the $\mathbb{C}$-linear map
$$f_{\mathbb{C}}(x) := \frac{1}{2}\big(f(x) - if(ix)\big) = \frac{1}{2i}\big(f(ix) + if(x)\big),$$
$x \in V$ and the $\mathbb{C}$-anti-linear map
$$f_{\overline{\mathbb{C}}}(x) := \frac{1}{2}\big(f(x) + if(ix)\big) = \frac{1}{2i}\big(-f(ix) + if(x)\big),$$
$x \in V$. The map $f_{\mathbb{C}}$ (resp. $f_{\overline{\mathbb{C}}}$) is called the $\mathbb{C}$-*linear* (resp. $\mathbb{C}$-*anti-linear*) *part*

of f. Then f is $\mathbb{C}$-linear if and only if $f = f_{\mathbb{C}}$, i.e. $f_{\overline{\mathbb{C}}} = 0$ and f is $\mathbb{C}$-antilinear if and only if $f = f_{\overline{\mathbb{C}}}$, i.e. $f_{\mathbb{C}} = 0$. Further, it follows that

$$\mathrm{Hom}_{\mathbb{R}}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W) + \mathrm{Hom}_{\overline{\mathbb{C}}}(V, W)$$

In particular, $\mathrm{Hom}_{\mathbb{R}}(V, W)$ is even a $\mathbb{C}$-subspace of W^V. Moreover,

$$\mathrm{Hom}_{\mathbb{C}}(V, W) \cap \mathrm{Hom}_{\overline{\mathbb{C}}}(V, W) = 0,$$

since $f(ix) = if(x) = -if(x)$ and so $f(x) = 0$ for every f in the intersection and for all $x \in V$.

This shows that in every representation $f = g + h$ with $g \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ and $h \in \mathrm{Hom}_{\overline{\mathbb{C}}}(V, W)$, it is necessary that $g = f_{\mathbb{C}}$ and $h = f_{\overline{\mathbb{C}}}$. For,

$$g - f_{\mathbb{C}} = f_{\overline{\mathbb{C}}} - h \in \mathrm{Hom}_{\mathbb{C}}(V, W) \cap \mathrm{Hom}_{\overline{\mathbb{C}}}(V, W) = 0$$

In the terminology of Example 3.B.13(1 therefore: $\mathrm{Hom}_{\mathbb{R}}(V, W)$ is the direct sum of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ and $\mathrm{Hom}_{\overline{\mathbb{C}}}(V, W)$, see also subsection 5.F. In the case $V = W = \mathbb{C}$, $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ and $\mathrm{Hom}_{\overline{\mathbb{C}}}(\mathbb{C}, \mathbb{C})$ are 1-dimensional over $\mathbb{C}$ with $\mathbb{C}$-bases the identity $z \longmapsto z$ and the complex conjugation $z \longmapsto \overline{z}$, respectively.

Therefore, $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ is 2-dimensional over $\mathbb{C}$ with $\mathbb{C}$-basis $z, \overline{z}$ and 4-dimensional over $\mathbb{R}$ with $\mathbb{R}$-basis $z, \overline{z}, iz, i\overline{z}$.