5.C. Space of Linear Maps

Let V and W be arbitrary vector spaces over a field K. Then the set $\mathrm{Hom}_K(V,W)$ of all K-homomorphisms from V into W is a subset of the K-vector space W^V of all maps from V into W. Analog of 5.A.12 is:

5.C.1 Theorem Let V and W be K-vector spaces. Then $\mathrm{Hom}_K(V,W)$ is a K-subspace of the K-vector space W^V of all maps from V into W, i.e. we have:

(1) The zero-map 0 belongs to $\mathrm{Hom}_K(V,W)$.

(2) If $f, g \in \mathrm{Hom}_K(V,W)$, then $f+g \in \mathrm{Hom}_K(V,W)$

(3) If $f \in \mathrm{Hom}_K(V,W)$, then $af \in \mathrm{Hom}_K(V,W)$ for every $a \in K$.

Proof (1) is trivial; the zero-map 0 is clearly K-linear.

(2) For all $a, b \in K$ and all $x, y \in V$, we have:

$$(f+g)(ax+by) = f(ax+by) + g(ax+by) =$$

$bf(x)+bf(y)+ag(x)+bg(y)$
$= b(f+g)(x) + c(f+g)(y)$,
i.e. $f+g$ is also K-linear.

(3) For all $b, c \in K$ and all $x, y \in V$, we have: $(af)(bx+cy) = af(bx+cy)$
$= a(bf(x)+cf(y)) \overset{*}{=} b(af)(x)+c(af)(y)$,
i.e. af is also K-linear. (Note here that for $*$ we have used the commutativity of multiplication in K.)

The similar simple proofs of the following computation-rules are left to the reader (see proof of 5.A.13):

5.C.2 Theorem Let U, V, W be vector spaces over a field K. For arbitrary homomorphisms $f, f': U \longrightarrow V$ and $g, g': V \longrightarrow W$ and all $a, b \in K$, we have:

(1) $f\circ(g+g') = f\circ g + f\circ g'$ and
$(f+f')\circ g = f\circ g + f'\circ g$.

(2) $(af)\circ(bg) = (ab)(f\circ g)$.

From 5.C.2 it follows that: for a K-vector space V, the set $\mathrm{End}_K V$ of all

K-endomorphisms of V is not only a K-vector space, but with the (vector space) addition and the composition as multiplication is also a ring with the identity id_V as the unit-element. Moreover, the ring multiplication and the scalar-multiplication on $\operatorname{End}_K V$ are compatible, see 5.C.2 (2). Such a situation occurs in many places. Therefore, we make the following general definition:

5.C.3 Definition Let K be a field and let A be a K-vector space, moreover, a multiplication $A \times A \longrightarrow A$ is defined. Then A is called a K-algebra if the following conditions are satisfied:

(1) A is with the vector space addition and the given multiplication is a ring.

(2) For all $a, b \in K$ and all $x, y \in A$, we have
$$(ax)(by) = (ab)(xy)$$

If A and B are two K-algebras, then a map $\varphi : A \longrightarrow B$ is called a <u>K-algebra homomorphism</u> if

(1) φ is a K-vector space homomorphism.

(2) φ is compatible with the multiplications on A and B, i.e.

$$\varphi(xy) = \varphi(x)\,\varphi(y) \text{ for all } x, y \in A,$$

and moreover, $\varphi(1_A) = 1_B$.

In particular, a K-algebra homomorphism is also a ring homomorphism.

From 5.C.1 and 5.C.2 it follows that:

5.C.4 Theorem Let K be a field and let V be a K-vector space. Then $\mathrm{End}_K V$ is a K-algebra. Moreover, the unit-group $(\mathrm{End}_K V)^\times$ of $\mathrm{End}_K V$ is the automorphism group $\mathrm{Aut}_K V$ of V.

5.C.5 Example (Function-Algebras)

An important class of (commutative) algebras is formed by the function-algebras. For an arbitrary field K and an arbitrary set D, the set K^D of all

K-valued functions on D is a commutative K-algebra in a natural way and the substitution maps (see Example 5.B.3(1) are K-algebra homomorphisms. All examples (of subspaces) in 1.C.10 (4), (5), (6) and in 1.C, Exercise 12, are subalgebras of algebras of the type K^D. -- A subset A' of an K-algebra A is called a K-subalgebra of A if A' is a K-subspace as well as a subring of A. However, the subspace $K[t]_n$ of $K[t]$ given in 1.C.10(6) is not a subalgebra of $K[t]$.

5.C.6 Example The field of complex numbers $\mathbb{C}$ is a 2-dimensional $\mathbb{R}$-algebra with the $\mathbb{R}$-basis $1, i$. It has only the trivial $\mathbb{R}$-subalgebras $\mathbb{R} = \mathbb{R}\cdot 1_{\mathbb{C}}$ and $\mathbb{C}$.

5.C.7 Example ($\mathbb{C}$-anti-linear maps) Let V and W be two complex vector-spaces (over $\mathbb{C}$), we consider them also as $\mathbb{R}$-vector spaces in a natural way, see Example 1.B.5. A map $h : V \longrightarrow W$ is called $\mathbb{C}$-antilinear (or $\mathbb{C}$-semi-linear) if h is additive and if

$$h(zx) = \bar{z}h(x)$$

for all $z \in \mathbb{C}$ and all $x \in V$. The set of all $\mathbb{C}$-anti-linear maps from V into W is denoted by $\mathrm{Hom}_{\overline{\mathbb{C}}}(V, W)$.

A map $h : V \longrightarrow W$ is clearly $\mathbb{C}$-linear (resp. $\mathbb{C}$-anti-linear) if and only if h is $\mathbb{R}$-linear and for all $x \in V$, we have $h(ix) = ih(x)$ (resp. $h(ix) = -ih(x)$).

Like the $\mathbb{C}$-linear maps, the $\mathbb{C}$-anti-linear maps from V into W, also form a $\mathbb{C}$-subspace of W^V.

Now, let $f : V \longrightarrow W$ be an arbitrary $\mathbb{R}$-linear map. Then

$$f = f_{\mathbb{C}} + f_{\overline{\mathbb{C}}}$$

with the $\mathbb{C}$-linear map

$$f_{\mathbb{C}}(x) := \frac{1}{2}\left(f(x) - if(ix)\right) = \frac{1}{2i}\left(f(ix) + if(x)\right),$$

$x \in V$ and the $\mathbb{C}$-anti-linear map

$$f_{\overline{\mathbb{C}}}(x) := \frac{1}{2}\left(f(x) + if(ix)\right) = \frac{1}{2i}\left(-f(ix) + if(x)\right),$$

$x \in V$. The map $f_{\mathbb{C}}$ (resp. $f_{\overline{\mathbb{C}}}$) is called the $\mathbb{C}$-linear (resp. $\mathbb{C}$-anti-linear) part of f. Then f is $\mathbb{C}$-linear if and only if $f = f_{\mathbb{C}}$, i.e. $f_{\overline{\mathbb{C}}} = 0$ and f is $\mathbb{C}$-antilinear if and only if $f = f_{\overline{\mathbb{C}}}$, i.e. $f_{\mathbb{C}} = 0$. Further, it follows that

$$\mathrm{Hom}_{\mathbb{R}}(V,W) = \mathrm{Hom}_{\mathbb{C}}(V,W) + \mathrm{Hom}_{\overline{\mathbb{C}}}(V,W).$$

In particular, $\mathrm{Hom}_{\mathbb{R}}(V,W)$ is even a $\mathbb{C}$-subspace of W^V. Moreover,

$$\mathrm{Hom}_{\mathbb{C}}(V,W) \cap \mathrm{Hom}_{\overline{\mathbb{C}}}(V,W) = 0,$$

since $f(ix) = if(x) = -if(x)$ and so $f(x) = 0$ for every f in the intersection and for all $x \in V$.

This shows that in every representation $f = g + h$ with $g \in \mathrm{Hom}_{\mathbb{C}}(V,W)$ and $h \in \mathrm{Hom}_{\overline{\mathbb{C}}}(V,W)$, it is necessary that $g = f_{\mathbb{C}}$ and $h = f_{\overline{\mathbb{C}}}$. For,

$$g - f_{\mathbb{C}} = f_{\overline{\mathbb{C}}} - h \in \mathrm{Hom}_{\mathbb{C}}(V,W) \cap \mathrm{Hom}_{\overline{\mathbb{C}}}(V,W) = 0$$

In the terminology of Example 3.B.13(v), therefore: $\mathrm{Hom}_{\mathbb{R}}(V,W)$ is the direct sum of $\mathrm{Hom}_{\mathbb{C}}(V,W)$ and $\mathrm{Hom}_{\overline{\mathbb{C}}}(V,W)$, see also subsection 5.F. In the case $V = W = \mathbb{C}$, $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$ and $\mathrm{Hom}_{\overline{\mathbb{C}}}(\mathbb{C},\mathbb{C})$ are 1-dimensional over $\mathbb{C}$ with $\mathbb{C}$-bases the identity $z \longmapsto z$ and the complex conjugation $z \longmapsto \bar{z}$, respectively. Therefore, $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ is 2-dimensional over $\mathbb{C}$ with $\mathbb{C}$-basis $z, \bar{z}$ and 4-dimensional over $\mathbb{R}$ with $\mathbb{R}$-basis $z, \bar{z}, iz, i\bar{z}$.