5.D Linear Maps and Bases

In this subsection we prove assertion on the existence of linear maps. We begin with two lemmata.

5.D.1 Lemma Let V and W be two K-vector spaces, x_i, $i \in I$, be a family of elements in V and let $f: V \longrightarrow W$ be a K-linear map. Then:

(1) If $f(x_i)$, $i \in I$, is a generating system for W, then f is surjective.

(2) If $f(x_i)$, $i \in I$, is linearly independent then x_i, $i \in I$, is linearly independent and the restriction of f to the subspace $U := \sum_{i \in I} Kx_i$ generated by x_i, $i \in I$, is injective.

(3) If x_i, $i \in I$, is a generating system for V and f is surjective, then $f(x_i)$, $i \in I$, is also a generating system for W.

(4) If x_i, $i \in I$, is linearly independent and f is injective, then $f(x_i)$, $i \in I$, is also linearly independent.

Proof (1) Let $y \in W$ and $y = \sum a_i f(x_i)$, $(a_i)_{i \in I} \in K^{(I)}$. Then $y = f\left(\sum_{i \in I} a_i x_i\right) \in \operatorname{Im} f$.

(2) Suppose that $\sum_{i \in I} a_i x_i = 0$, $(a_i) \in K^{(I)}$. Then $0 = f\left(\sum_{i \in I} a_i x_i\right) = \sum_{i \in I} a_i f(x_i)$ and hence $a_i = 0$ for all $i \in I$. Suppose now $f\left(\sum_{i \in I} a_i x_i\right) = 0$, $(a_i) \in K^{(I)}$. Then $0 = f\left(\sum_{i \in I} a_i x_i\right) = \sum_{i \in I} a_i f(x_i)$. Then again $a_i = 0$ for all $i \in I$ and so $\sum_{i \in I} a_i x_i = 0$.

This also proves the injectivity of the restriction $f \Big| \sum_{i \in I} K x_i$.

(3) Let $y \in W$ and $y = f(x)$ with $x = \sum_{i \in I} a_i x_i \in V$, $(a_i) \in K^{(I)}$. Then $y = f(x) = f\left(\sum_{i \in I} a_i x_i\right) = \sum_{i \in I} a_i f(x_i)$ is a linear combination of $f(x_i)$, $i \in I$.

(4) Suppose that $\sum_{i \in I} a_i f(x_i) = 0$ for some $(a_i) \in K^{(I)}$. Then $0 = \sum_{i \in I} a_i f(x_i) = f\left(\sum_{i \in I} a_i x_i\right)$

and hence $\sum_{i \in I} a_i x_i = 0$ by the injectivity of f, but then $a_i = 0$ for all $i \in I$, since x_i, $i \in I$, is linearly independent.

5.D.2 Lemma Let V, W be K-vector spaces and let $f, g : V \longrightarrow W$ be K-linear maps. Then the set $x \in V$ with $f(x) = g(x)$ form a K-subspace of V. In particular, $f = g$ if and only if $f(x_i) = g(x_i)$ for all the elements of a generating system x_i, $i \in I$, of V.

Proof $\{x \in V \mid f(x) = g(x)\} = \mathrm{Ker}(f - g)$ is a K-subspace of the K-vector space V.

A complete overview on the K-linear maps $f : V \longrightarrow W$ if the values $f(v_i)$, $i \in I$, are given on a basis v_i, $i \in I$, of V:

5.D.3 Theorem Let V, W be K-vector spaces. Let v_i, $i \in I$, be a basis of V and let w_i, $i \in I$, be an arbitrary family of elements in W. Then there exists a unique K-linear map such that $f(v_i) = w_i$ for all $i \in I$ and moreover,

$$f(x) = \sum_{i\in I} a_i w_i \text{ for } x = \sum_{i \in I} a_i v_i \in V, \quad (a_i)_{i\in I} \in K^{(I)}.$$

For this map f we have:

(1) f is surjective if and only if $w_i, i \in I$, is a generating system for W.

(2) f is injective if and only if $w_i, i \in I$, is linearly independent.

(3) f is bijective, i.e. an isomorphism if and only if $w_i, i \in I$, is a basis of W.

Proof See (*) on Page 5D/10.

5.D.4 Example With the notation and assumptions Theorem 5.D.3, in particular, shows that the map

$$\mathrm{Hom}_K(V, W) \longrightarrow W^I, \quad f \mapsto (f(v_i))_{i \in I}$$

is bijective. Obviously, this map is also K-linear and hence it is an isomorphism of K-vector spaces.

From Theorem 5.D.3 it follows immediately that:

5.D.5 Theorem Two finite dimensional

K-vector spaces are isomorphic if and only if they have equal dimensions. In particular, every K-vector space of dimension $n \in \mathbb{N}$ is isomorphic to K^n.

The assertion 3.A.4 is a special case of 5.D.3: For an arbitrary family $v_i, i \in I$, in a K-vector space V, the map

$$K^{(I)} \longrightarrow V, \quad (a_i)_{i \in I} \longmapsto \sum_{i \in I} a_i v_i$$

is the uniquely determined K-linear map $K^{(I)} \longrightarrow V$ with $e_i \longmapsto v_i$, $i \in I$. It is an isomorphism if and only if v_i, $i \in I$, is a basis of V. In particular, the isomorphisms $K^n \longrightarrow V$ for an n-dimensional K-vector space V are precisely the maps $(a_1, \ldots, a_n) \longmapsto a_1 v_1 + \cdots + a_n v_n$, where $v_1, \ldots, v_n$ is a basis of V. The inverse isomorphisms $V \longrightarrow K^n$ are the maps

$$x \longmapsto \left(v_1^*(x), \ldots, v_n^*(x)\right),$$

where $v_1^*, \ldots, v_n^*$ are the corresponding

(linear) coordinate functions $V \to K$.
Therefore, in general elements of a K-vector space V can be identified with I-tuples $(a_i) \in K^{(I)}$, once we choose a distinguished basis v_i, $i \in I$ of V.
The choice of a basis of V is also called a guage of V.

The following important theorem says that for two finite-equal dimensional K-vector spaces both the conditions "injective" (resp. "surjective") are enough to verify for an isomorphism:

5.D.6 Theorem Let V and W be finite dimensional K-vector spaces of equal dimensions. For a K-linear map $f : V \longrightarrow W$, the following statements are equivalent:

(1) f is bijective, i.e. an isomorphism.
(2) f is injective.
(2′) There exists a K-linear map $g : W \to V$ such that $g \circ f = id_V$.
(3) f is surjective.
(3′) There exists a K-linear map $h : W \to V$

such that $f \circ h = id_W$.

Proof Among the equivalences

$(1) \Rightarrow (2') \Rightarrow (2) \Rightarrow (1)$ and

$(1) \Rightarrow (3') \Rightarrow (3) \Rightarrow (1)$, the only

$(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$ are not trivial.

$(2) \Rightarrow (1)$: Suppose that f is injective and $v_1, \ldots, v_n$ is a basis of V. By 5.D.1(1) the vectors $f(v_1), \ldots, f(v_n)$ are linearly independent and hence is a basis of W, since $\mathrm{Dim}_K W = \mathrm{Dim}_K V = n$. Therefore f is an isomorphism by 5.D.3 (3).

$(3) \Rightarrow (1)$: Suppose that f is surjective and $v_1, \ldots, v_n$ is a basis of V. By 5.D.1(3) the vectors $f(v_1), \ldots, f(v_n)$ is a generating system for W and hence is a basis of W, since $\mathrm{Dim}_K W = \mathrm{Dim}_K V = n$. Therefore f is an isomorphism by 5.D.3 (3).

Note that 5.D.6 is applicable to a linear operator f on a finite dimensional K-vector space V.

5.D.7 Example Let $f: V \longrightarrow W$ be a K-linear map of finite-equal dimensional K-vector spaces. By 5.D.6 f is surjective if and only if f is injective, i.e.: The equation $f(x)=y$ has a solution $x \in V$ for every $y \in W$ if and only if the homogeneous equation $f(x)=0$ has only trivial solution $x=0$. We note the following two simple applications:

(1) A system of linear equations over K

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

with as many as unknowns as the number of equations has a solution for each choice of $b_1, \ldots, b_n$ if and only if the corresponding homogeneous system of linear equations has only trivial solution! Moreover, in this case, for arbitrary $b_1, \ldots, b_n$ this system of equations has exactly one solution.

(2) We prove the theorem on the Hermite-interpolation:
Let $a_1, \dots, a_r \in \mathbb{K}$ be pairwise distinct and let $n_1, \dots, n_r$ be positive natural numbers with $m := n_1 + \dots + n_r$.
Then the $\mathbb{K}$-linear map

$$H : \mathbb{K}[t]_m \longmapsto \mathbb{K}^{n_1} \times \dots \times \mathbb{K}^{n_r}$$

which maps every polynomial function g of degree $< m$ to the number-tuple of the derivatives

$$g^{(0)}(a_1), \dots, g^{(n_1-1)}(a_1), \dots, g^{(0)}(a_r), \dots, g^{(n_r-1)}(a_r)$$

is bijective. By the above equivalence in 5.D.6, it is enough to show that this map is injective, i.e. if g is a polynomial of degree $< m$ such that all the derivatives of g given above vanish, then $g = 0$. If all the derivatives given above vanish, then g has zeros at the points a_i of order $\geq n_i$, $i = 1, \dots, r$.
Then $g = q(t-a_1)^{n_1} \cdots (t-a_r)^{n_r}$ with a polynomial function $q \in \mathbb{K}[t]$.
But then $m > \deg g = \deg q + n_1 + \dots + n_r = \deg q + m$ if $q \neq 0$. This proves that $q = 0$

and hence $g=0$.

Therefore the uniqueness-assertion already implies the existence-assertion in the theorem on the Hermite-interpolation. This theorem can also be formulated for an arbitrary ground field K, see Exercise 11 of subsection 10.A.

5.D.8 Remark. An analogous assertion to 5.D.6 does not hold for infinite dimensional K-vector spaces, see Exercise 10.

Insert the following proof at (*) on Page 5D/4:

Proof The uniqueness of f follows from 5.D.2. The given map f is clearly K-linear with $f(v_i) = w_i$, $i \in I$.
(1) follows from 5.D.1 (1) and 5.D.1(3).
(2) follows from 5.D.1 (2) and 5.D.1(4).
(3) is a combination of (1) and (2).