5.E The Rank Theorem

For the basic investigation of linear maps between finite dimensional vector spaces, the following Rank-Theorem is very important:

5.E.1 Rank-Theorem Let $f: V \longrightarrow W$ be a K-linear map from a finite dimensional K-vector space into an arbitrary K-vector space W. Then the subspaces $\operatorname{Ker} f \subseteq V$ and $\operatorname{Im} f \subseteq W$ are also finite dimensional and we have:

$$\operatorname{Dim}_K V = \operatorname{Dim}_K \operatorname{Ker} f + \operatorname{Dim}_K \operatorname{Im} f.$$

This assertion motivates the following

5.E.2 Definition For a K-linear map $f: V \longrightarrow W$ of K-vector spaces V and W, the dimension of the image $\operatorname{Im} f$ of f is called the rank of f and is denoted by $\operatorname{Rank} f = \operatorname{Rank}_K f$.

With this definition the Rank-Theorem states that: In the situation of 5.E.1,

$\operatorname{Dim}_K V = \operatorname{Dim}_K \operatorname{Ker} f + \operatorname{Rank} f$ or

$$\operatorname{Rank} f = \operatorname{Codim}_K(\operatorname{Ker} f, V).$$

<u>Proof of 5.E.1.</u> $\operatorname{Ker} f$ is a subspace of a finite dimensional K-vector space V and hence is finite dimensional. Further, it has a basis $u_1, \dots, u_r$ ($\in \operatorname{Ker} f$). Let $v_1, \dots, v_n \in V$ be a K-basis of V. Then the image $f(V)$ of f is generated by $f(v_1), \dots, f(v_n)$; in particular, $f(V)$ is finite dimensional. Further, the generating system $f(v_1), \dots, f(v_n)$ contains a K-basis of $f(V)$; by renumbering we may assume that $f(v_1), \dots, f(v_s)$ is a basis of $\operatorname{Im} f = f(V)$.

We shall show that $u_1, \dots, u_r, v_1, \dots, v_s$ is a basis of V; with this the Rank-Theorem will be proved.

For linear independence, suppose that

$$a_1 u_1 + \cdots + a_r u_r + b_1 v_1 + \cdots + b_s v_s = 0,$$

$a_1, \dots, a_r, b_1, \dots, b_s \in K$. Applying f we get:

$$0 = a_1 f(u_1) + \cdots + a_r f(u_r) + b_1 f(v_1) + \cdots + b_s f(v_s)$$
$$= b_1 f(v_1) + \cdots + b_s f(v_s).$$

From the linear independence of $f(v_1), \dots, f(v_s)$, it follows that $b_1 = \cdots = b_s = 0$.

Therefore $a_1u_1+\cdots+a_ru_r=0$ and from the linear independence, we get $a_1=\cdots=a_r=0$.

Now, to show that $u_1,\dots,u_r,v_1,\dots,v_s$ is a generating system for V. Let $x\in V$. Then $f(x)=b_1f(v_1)+\cdots+b_sf(v_s)$ for some $b_1,\dots,b_s\in K$ and it follows that $x-(b_1v_1+\cdots+b_sv_s)\in \operatorname{Ker} f$. Therefore there exist elements $a_1,\dots,a_r\in K$ such that

$$x-(b_1v_1+\cdots+b_sv_s)=a_1u_1+\cdots+a_ru_r$$

and x is a K-linear combination of $u_1,\dots,u_r,v_1,\dots,v_s$.

5.E.3 Example (Dimension-Formula)

We shall prove once more the Dimension-Formula 3.B.11. Let U and W be finite dimensional subspaces of the K-vector space V. Then the map

$$f: U\oplus W \longrightarrow V,\ (x,y)\longmapsto x-y$$

is K-linear with image $\operatorname{Im} f = U+W$ the sum of U and W. Further, the kernel of f is clearly the subspace $X:=\{(x,x)\,/\,x\in U\cap W

Note that X is isomorphic to $U \cap W$ with an isomorphism $(x, x) \longrightarrow x$. In particular, $\mathrm{Dim}_K X = \mathrm{Dim}_K U \cap W$.

By Rank-Theorem we get:

$$\mathrm{Dim}_K U + \mathrm{Dim}_K W = \mathrm{Dim}_K (U \oplus W)$$
$$= \mathrm{Dim}_K \mathrm{Ker} f + \mathrm{Dim}_K \mathrm{Im} f$$
$$= \mathrm{Dim}_K U \cap W + \mathrm{Dim}_K (U + W).$$

5.E.4 Example The equivalence of statements (1), (2), (3) in Theorem 5.D.6 immediately follows from 5.E.1. For, with the notation of 5.D.6, if f is injective, i.e. $\mathrm{Ker} f = 0$, then $\mathrm{Dim}_K \mathrm{Im} f = \mathrm{Dim}_K V = \mathrm{Dim}_K W$ and hence $\mathrm{Im} f = W$. Analogously, from the surjectivity of f, the injectivity of f follows.

5.E.5 Example (Rank of system of linear equations) Let

$$a_{11} x_1 + \cdots + a_{1n} x_n = b_1$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$a_{m1} x_1 + \cdots + a_{mn} x_n = b_m$$

be a system of m linear equations in n unknowns over a field K. As in Example 5.B.4, we use the linear map $f: K^n \longrightarrow K^m$ with

$$f(e_j) := y_j := (a_{1j}, \dots, a_{mj})$$

$j = 1, \dots, n$ (where $e_1, \dots, e_n \in K^n$ is the standard basis of K^n), to denote the given system of linear equations. We are then looking for all $x = (x_1, \dots, x_n) \in$ such that $f(x) = b := (b_1, \dots, b_m)$.

The rank of f is the dimension of the subspace of K^m generated by $y_1, \dots, y_m$. This rank is also called the <u>rank of the (system of) linear equations</u>. The solution space L_0 of the correspondin system of homogeneous linear equation is the kernel of f. By the Rank-Theorem, we have:

$$\mathrm{Dim}_K L_0 = n - \mathrm{Rank}\, f.$$

If the given inhomogeneous system of equations has a solution x', i.e. $b \in \mathrm{Im}\, f$, then its solution space is equal to the affine subspace $L = x' + L_0$ of K^n. Further, since by definition, the dimension of L is the dimension of L_0, it follows that:

5.E.6 *The solution space of a system of linear equations of rank r in n unknowns over a field K, is an affine subspace of K^n of dimension $n-r$, in case the given system of linear equations has at least one solution.*