5.F Direct sums and Projections

Let V be a K-vector space and let U_i, $i \in I$, be a family of K-subspaces. Then the map $\bigoplus_{i\in I} U_i \longrightarrow V$ from the (external) direct sum of U_i, $i\in I$, into V defined by $(u_i)_{i\in I} \longmapsto \sum_{i\in I} u_i$ is a K-homomorphism, its image is the sum $\sum_{i\in I} U_i$ of U_i, $i\in I$, in V. Moreover, if this map is injective, then it induces an isomorphism of $\bigoplus_{i\in I} U_i$ onto $\sum_{i\in I} U_i$. In this case we say that the sum $\sum_{i\in I} U_i$ of the subspaces U_i, $i\in I$, is a <u>direct sum</u> and this subspace of V is denoted by $\sum_{i\in I}^{\oplus} U_i$ or occasionally by $\bigoplus_{i\in I} U_i$ if there is no confusion with the external direct sum of U_i, $i\in I$. If the sum of finitely many subspaces $U_1, \dots, U_n$ of V is direct, then for $U_1 + \cdots + U_n$ we also write $U_1 \oplus \cdots \oplus U_n = \bigoplus_{i=1}^{n} U_i$.

5.F.1 Example Let x_i, $i\in I$, be a basis of the K-vector space V. Then clearly V is the direct sum of the (one dimensional) K-subspaces $U_i := Kx_i$, $i\in I$. Therefore the concept of direct sum generalise the concept of the basis.

If V has a representation of the direct sum $V = \sum_{i\in I}^{\oplus} U_i$ of subspaces U_i, $i\in I$, and for each summand U_i, $i\in I$, there is a basis y_j, $j\in J_i$ with pairwise disjoint subsets J_i, $i\in I$, then all these bases together y_j, $j\in \biguplus_{i\in I} J_i$ form a basis of V. The canonical isomorphism of the external direct sum onto the inner direct sum maps the basis of the external direct sum formed by using the bases y_j, $j\in J_i$, $i\in I$, onto the basis y_j, $j\in \biguplus_{i\in I} J_i$ of the internal direct sum. Therefore a direct sum decomposition is a first step of a finer decomposition of V with the help of a basis.

5.F.2 If V is a direct sum of subspaces $U_1,\dots,U_n$, then V is finite dimensional if and only if the subspaces U_i, $i=1,\dots,n$, are finite dimensional. Moreover, in this case $\operatorname{Dim}_K V = \sum_{i=1}^{n} \operatorname{Dim}_K U_i$.

The following characterisation of direct sums shows the analogy of the computations with direct sums with the computation with bases (resp. with linear independent families)

5.F.3 Theorem Let U_i, $i\in I$, be a family of subspaces of the K-vector space V. Then the following statements are equivalent:

(1) The sum of the U_i, $i\in I$, is direct, i.e. $\sum_{i\in I} U_i = \sum_{i\in I}^{\oplus} U_i$

(2) For every $u \in \sum_{i\in I} U_i$ there exist unique elements $u_i \in U_i$, $i \in I$, almost all are 0 with $u = \sum_{i\in I} u_i$.

(2') If $0 = \sum_{i\in I} u_i$ with elements $u_i \in U_i$, $i \in I$, almost all are 0, then $u_i = 0$ for all $i \in I$.

(3) For every $j \in I$, $U_j \cap \left(\sum_{i\in I\setminus\{j\}} U_i\right) = 0$.

Proof The equivalence of (1), (2) and (2') is immediate from the definitions.

(2') $\Rightarrow$ (3) Let $x \in U_j \cap \left(\sum_{i\in I, i\neq j} U_i\right)$ then $x = \sum_{i\in I, i\neq j} u_i$ with $u_i \in U_i$, $i \in I, i\neq j$, almost all are 0 and so $0 = (-x) + \sum_{i\in I, i\neq j} u_i$. Since $-x \in U_j$, it follows from (2') that $-x = 0$, i.e. $x = 0$.

(3) $\Rightarrow$ (2') Suppose that $0 = \sum_{i\in I} u_i$, $u_i \in U_i$, $i \in I$, $u_i = 0$ for almost all $i \in I$. For a fixed $j \in I$, we have $u_j = \sum_{i\in I, i\neq j} (-u_i) \in U_j \cap \left(\sum_{i\in I, i\neq j} U_i\right)$ and hence $u_j = 0$.

In the representation $u = \sum_{i\in I} u_i$ of $u \in \sum_{i\in I}^{\oplus} U_i =: U$ in 5.F.3 (2) the u_i, $i \in I$, are called the components of u with respect to the direct sum decomposition of $U = \sum_{i\in I}^{\oplus} U_i$. The special case of a family with two subspaces is noted as corollary to 5.F.3:

5.F.4 Corollary Let U, W be subspaces of a K-vector space V. Then the sum $U+W$ of the subspaces U and W is direct if and only if $U \cap W = 0$.

If there is a representation of V as a direct sum $V = U \oplus W$ of subspaces U and W of V, then W is called a complement of U in V (and U is a complement of W in V). Every subspace of a vector space has a complement (however, other than trivial subspaces complement is not uniquely determined, see Exercises 7.3 and T7.19

5.F.5 Theorem Let V be a finite dimensional K-vector space. Then every subspace U of V has a complement in V.

Proof Let $u_1, \dots, u_r \in U$ be a basis of U. Extend $u_1, \dots, u_r$ to a basis $u_1, \dots, u_r, w_1, \dots, w_s$ of V. Then the subspace $W := Kw_1 + \cdots + Kw_s$ is clearly a complement of U in V.

5.F.6 Remark Since in an arbitrary vector space every linearly independent family can be extended to a basis, the assertion in 5.F.5 also holds even if V is not finite dimensional.

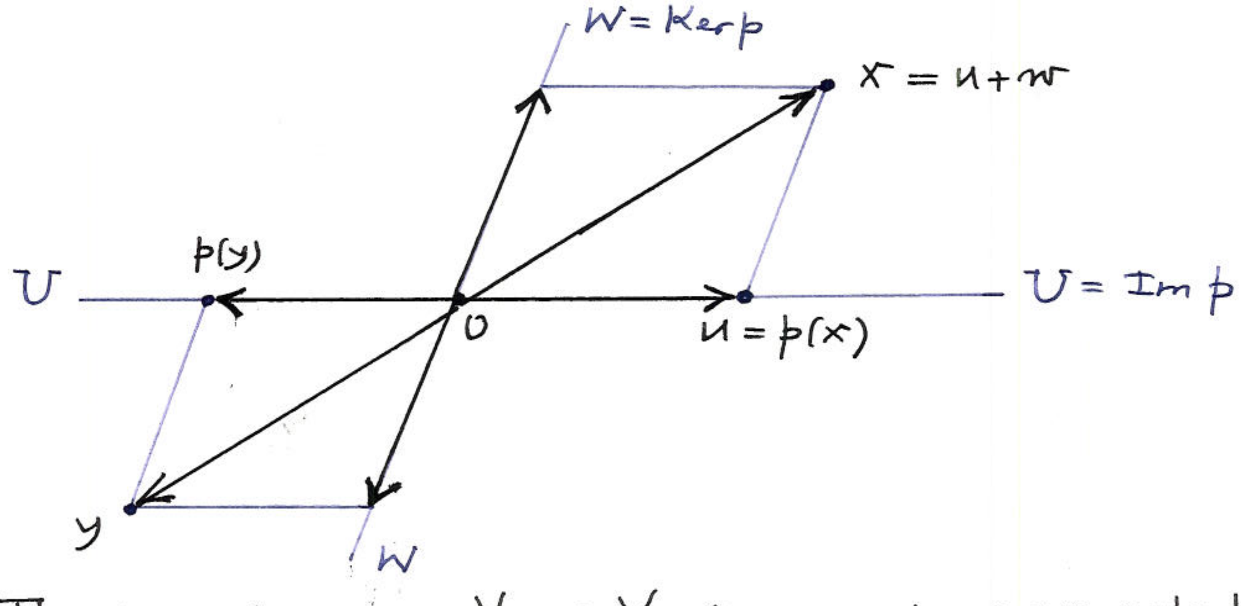
Let $V = U \oplus W$ be the direct sum of the subspaces U and W. Then by 5.F.3 (2) every element $x \in V$ has a unique decomposition $x = u + w$ with vectors $u \in U$, $w \in W$. Therefore the map

$$p : V \longrightarrow V, \quad x = u + w \longmapsto u$$

is well-defined. Clearly p is K-linear. Further, $\operatorname{Im} p = U$ and $\operatorname{Ker} p = W$. Moreover, p^2 $(= p \circ p) = p$. Therefore p is a projection in the sense of the following definition:

5.F.7 Definition A linear operator $p : V \longrightarrow V$ on a K-vector space V is called a projection (of V) if $p^2 = p$.

The projections of V are therefore idempotent elements in the K-algebra $\operatorname{End}_K V$ of the set of all K-endomorphisms of V. The above constructed projection p of $V = U \oplus W$ is called the projection onto U along W. Note that $U = \operatorname{Im} p = \{x \in V \mid p(x) = x\} = \operatorname{Ker}(\mathrm{id}_V - p)$



The projection $q : V \longrightarrow V$, $x = u + w \longmapsto w$ onto W along U is then simply the map $\mathrm{id}_V - p$. Both these projections p and $q = \mathrm{id}_V - p$ are called complementary

Conversely, every projection of V supply a direct sum decomposition of V:

5.F.8 Theorem *Let p be a projection of the K-vector space V. Then there is the direct sum decomposition $V = \operatorname{Im} p \oplus \operatorname{Ker} p$ of V and p is the projection onto $\operatorname{Im} p$ along $\operatorname{Ker} p$.*

Proof We need to show $\operatorname{Im} p + \operatorname{Ker} p = V$ and $\operatorname{Im} p \cap \operatorname{Ker} p = 0$. First, suppose $x \in V$. Then $x = p(x) + (x - p(x))$ with $p(x) \in \operatorname{Im} p$ and $x - p(x) \in \operatorname{Ker} p$, since $p(x - p(x)) = p(x) - p^2(x) = p(x) - p(x) = 0$. Now, if $x \in \operatorname{Im} p \cap \operatorname{Ker} p$. Then $x = p(y)$ with $y \in V$ and $p(x) = 0$. But then $x = p(y) = p^2(y) = p(p(y)) = p(x) = 0$. The last assertion follows from $x = p(x) + (x - p(x))$.

An arbitrary direct sum decomposition of V can also be described with the help of projections. Let V be the direct sum of the subspaces $U_i, i \in I$. For every $j \in I$, the map $p_j : V \longrightarrow V, x = \sum_{i \in I} x_i \longmapsto x_j$ which maps every vector $x \in V$ to its j-th component $x_j \in U_j$ with respect to the decomposition $V = \sum_{i \in I}^{\oplus} U_i$, is well-defined. Clearly p_j is the projection onto U_j along $\sum_{i \in I, i \neq j} U_i$. Further, the family $p_i, i \in I$, so defined satisfies the following properties:

(1) $\sum_{i \in I} p_i = \mathrm{id}_V$ (2) $p_i p_j = 0$ for all $i, j \in I, i \neq j$.

Note that the sum $\sum_{i\in I} p_i$ (even if infinitely many summands are $\neq 0$) makes sense, since for every fixed $x \in V$, $(\sum_{i\in I} p_i)(x) = \sum_{i\in I} p_i(x)$ has only finitely many non-zero summands. We say that an arbitrary family f_i, $i \in I$, of operators on V is <u>summable</u> if for every $x \in V$, $f_i(x) = 0$ for almost all $i \in I$ and in this case we use the notation $\sum_{i\in I} f_i$ as above.

More generally,

5.F.9 Theorem *Let p_i, $i \in I$, be a summable family of projections of the K-vector space. Suppose that p_i, $i \in I$, satisfy the above conditions (1) and (2). Then $V = \sum_{i\in I}^{\oplus} \operatorname{Im} p_i$ is the direct sum of the subspaces $\operatorname{Im} p_i$, $i \in I$, and p_i, $i \in I$, is the family of the corresponding projections.*

Proof Since $x = \mathrm{id}_V(x) = (\sum_{i\in I} p_i)(x) = \sum_{i\in I} p_i(x)$, $x \in V$, V is the sum of the subspaces $\operatorname{Im} p_i$, $i \in I$. Now, if $x \in (\operatorname{Im} p_j) \cap (\sum_{i\in I, i\neq j} \operatorname{Im} p_i)$. Then $x = p_j(y) = \sum_{i\in I, i\neq j} p_i(y_i)$ with $y, y_i \in V$, $i \in I$, and hence $x = p_j(y) = p_j^2(y) = p_j(\sum_{i\in I, i\neq j} p_i(y_i)) = \sum_{i\in I, i\neq j} p_j p_i(x) = 0$. Therefore the sum $V = \sum_{i\in I} \operatorname{Im} p_i$ is direct.