5.G Dual Spaces

For the computation with vectors in a K-vector space V, the linear forms $V \longrightarrow K$ on V play a decisive roll. This is also the case when V is not finite dimensional; in this case we use the existence of bases, see remark in 3.A.17.

5.G.1 Definition Let V be a vector space over a field K. Then the K-vector space

$$V^* = \mathrm{Hom}_K(V, K)$$

of the K-linear forms on V is called the *dual space of V.*

Clearly, V^* is a subspace of the space K^V of all K-valued functions on V. The vector space structure on V^* is

$$(af+bg)(x) = af(x) + bg(x),$$

$f, g \in V^*$; $a, b \in K$; $x \in V$.

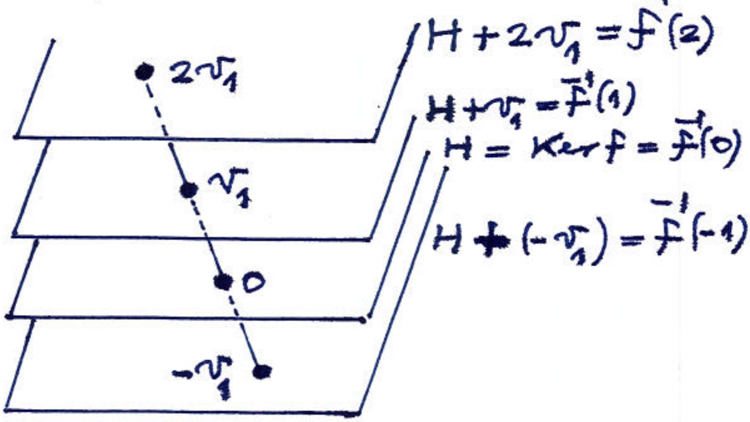
5.G.2 Example It is very important to obtain a geometric representation of the linear forms on a K-vector space V. Let $f: V \longrightarrow K$ be a non-zero linear form on V. Then f is clearly surjective. Further, let $H = \operatorname{Ker} f \ (\neq V)$. Then for every vector $v \in V \setminus H$,

$$V = H \oplus Kv.$$

For, if $x \in V$, then $x - \dfrac{f(x)\,v}{f(v)} \in H$

and hence $x \in H + Kv$.

Such a subspace H of V is called a hyperplane in V, see also Exercise 1. If V is finite dimensional, then the hyperplanes in V are precisely the 1-codimensional ($\operatorname{Dim}_K V - 1$ dimensional) subspaces of V. Clearly f is uniquely determined by the hyperplane H and the value of f on a single vector $v \in V$, which is not in H. In particular, f is completely determined by the affine hyperplane $\overset{-1}{f}(1) = H + v_1$ parallel to H, where v_1 is an arbitrary vector in V with $f(v_1) = 1$. The fibres of f are the parallel affine hyperplanes $H + v \subseteq V$, $v \in V$.



If V has additional structure, then the vector $v_1 \in V$ can often be chosen canonically, for example, if V has a scalar product, then v_1 can be chosen orthogonal to H. With this we get a close connection between the linear forms on V and the vectors itself in V. However, the vectors and linear forms are distinguished.

More detailed discussion on the duality will be given later in Chapter V. One can also compare this with the concept of the gradient in subsection 12 B.

5.G.3 Example Let K be a field and $n \in \mathbb{N}$. The linear forms on K^n are precisely the functions

$$f : K^n \longrightarrow K, \quad (x_1, \dots, x_n) \longmapsto a_1 x_1 + \dots + a_n x_n$$

with fixed $a_1, \dots, a_n \in K$ and indeed, $a_i = f(e_i)$ the value of f on the i-th vector e_i in the standard basis of K^n, $i = 1, 2, \dots, n$. If $f \neq 0$, then (f corresponds to) the kernel of f is the hyperplane of the solutions of the homogeneous linear equation

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0.$$

The assignment

$$f \longmapsto (a_1, \dots, a_n) = (f(e_1), \dots, f(e_n))$$

is an isomorphism of the dual space $(K^n)^*$ onto K^n, see Example 5.D.4. This allows us to identify f with the vector $(a_1, \dots, a_n) \in K^n$. This is an example of: how one can identify the linear forms on a vector space even with the vectors of this vector space, see Example 5.G.2.

For finite dimensional vector space V with the help of a basis of V, one can easily give a basis of V^*.

5.G.4 Theorem Let V be a finite dimensional K-vector space with basis v_i, $i \in I$. Then the family v_i^*, $i \in I$, of coordinate functions with respect to this basis is a basis of the dual basis V^* of V. In particular, $\operatorname{Dim}_K V = \operatorname{Dim}_K V^*$.

Proof Linear independence of v_i^*, $i \in I$:
From $0 = \sum_{i \in I} a_i v_i^*$, it follows that

$$0 = \left(\sum_{i \in I} a_i v_i^*\right)(v_j) = a_j \text{ for every } j \in I.$$

For an arbitrary $e \in V^*$, we have

$$\left(\sum_{i \in I} e(v_i) v_i^*\right)(v_j) = e(v_j) \text{ for all } j \in I$$

and hence by 5.D.2

$$e = \sum_{i \in I} e(v_i) \cdot v_i^* ;$$

Therefore, the family v_i^*, $i \in I$, generates the dual space V^*.

5.G.5 Definition Let v_i, $i \in I$, be a basis of the finite dimensional K-vector space V. Then the basis v_i^*, $i \in I$, of the coordinate functions with respect to v_i, $i \in I$, is called the dual basis with respect to the basis v_i, $i \in I$.

The dual basis v_i^*, $i \in I$, is, therefore, characterised by the equations:

$$v_i^*(v_j) = \delta_{ij}, \quad i, j \in I.$$

In the situation of the above definition the assignment $v_i \longmapsto v_i^*$, $i \in I$, defines an isomorphism of V onto V^*, but it essentially depends on the choice of the finite basis v_i, $i \in I$ of V. We further remark here that every single element of the dual basis v_i^*, $i \in I$, depends on the totality of the basis elements v_i, $i \in I$.

In the case of the standard basis e_j, $j \in I$, of $K^I = K^{(I)}$, e_i^* is the i-th canonical projection $K^I \longrightarrow K$ with $(a_j)_{j \in I} \longmapsto a_i$.

5.G.6 Remark If v_i, $i \in I$, is an infinite basis of V, then the coordinate functions v_i^*, $i \in I$, are in any case are linearly independent, but never forms a basis of V^*, cf. Exercise 2a). In this case, V and V^* are not even isomorphic (we have mentioned this without proof. If I is countably infinite, e.g. $I = \mathbb{N}$, i.e. $\operatorname{Dim}_K V = \aleph_0$, then the proof is indeed very simple: As remarked in 5.D.4, we have $V^* = \operatorname{Hom}_K(V, K) \cong K^{\mathbb{N}}$ and the space of sequences $K^{\mathbb{N}}$ never have a countable basis, see Exercise 3.A, 20a).

For establishing the relation between subspaces of V and V^*, first we prove:

5.G.7 Theorem *Let U be a subspace of the finite dimensional K-vector space V and let $x \in V$ be an arbitrary vector. Then $x \in U$ if and only if every linear form $e \in V^*$ which vanish on U also vanish on x. This assertion also holds even if V is not finite dimensional.*

Proof It is enough to show that: if $x \notin U$, then there exists a linear form $e \in V^*$ such that $U \subseteq \operatorname{Ker} e$ and $e(x) \neq 0$. The vector $x \notin U$ and a basis v_i, $i \in I$, of U together are linearly independent and therefore can be extended to a basis of V (this is also true in the case that V is not finite dimensional, see 3.A.17). Then for e one can choose any linear form which vanishes on v_i, $i \in I$, and $e(x) \neq 0$.

From 5.G.7 we get the following possibility to describe subspaces using linear forms

5.G.8 Corollary Let U_1 and U_2 be subspaces of the K-vector space V. Then $U_1 \subseteq U_2$ if and only if every linear form which vanish on U_2 also vanishes on U_1. In particular, $U_1 = U_2$ if and only if a linear form $e \in V^*$ vanish on U_1 then and only then if vanish on U_2.

In connection with the earlier assertions the following notation is useful:

Let V be a K-vector space. For a subspace U of V, let

$$U^\circ := \{ e \in V^* \,/\, e(x) = 0 \text{ for all } x \in U \}$$

and for a subspace W of V^*, let

$$^\circ W := \{ x \in V \,/\, e(x) = 0 \text{ for all } e \in W \}.$$

Then clearly U° and $^\circ W$ are subspaces of V^* respectively V. Moreover, if $U_1 \subseteq U_2 \subseteq V$, then $U_1^\circ \supseteq U_2^\circ$ and if $W_1 \subseteq W_2 \subseteq V^*$, then $^\circ W_1 \supseteq {}^\circ W_2$. Further, $U \subseteq {}^\circ(U^\circ)$ for every subspace U of V and $W \subseteq ({}^\circ W)^\circ$ for every subspace W of V^*.

Therefore $U^\circ \subseteq ({}^\circ(U^\circ))^\circ \subseteq U^\circ$ and hence $U^\circ = ({}^\circ(U^\circ))^\circ$.

Correspondingly $^\circ W \subseteq {}^\circ(({}^\circ W)^\circ)$.

Theorem 5.G.7 says that even

$$^\circ(U^\circ) = U$$

for every subspace U of V. However, The analogous equality $({}^\circ W)^\circ = W$ for every subspace

W of V^* does not hold in general, for example, see Exercise 2b). It holds if V is finite dimensional (or more generally, if W is a finite dimensional subspace of V^*. For this first we prove:

5.G.9 Theorem Let V be a finite dimensional K-vector space and let $W \subseteq V^*$ be a subspace of V^*. Then:

$$\operatorname{Dim} W = \operatorname{Codim}({}^{\circ}W, V) \; (= \operatorname{Dim} V - \operatorname{Dim} {}^{\circ}W).$$

Proof Let $f_1, \ldots, f_r$ be a basis of W. Then ${}^{\circ}W = H_1 \cap \cdots \cap H_r$, where $H_i = \operatorname{Ker} f_i$, $i = 1, \ldots, r$ and hence by the Codimension-formula 3.B.13 (2)

$$\operatorname{Codim} {}^{\circ}W \leq \sum_{i=1}^{r} \operatorname{Codim} H_i = r = \operatorname{Dim} W.$$

Conversely, suppose that $v_{s+1}, \ldots, v_n \in V$ is a basis of ${}^{\circ}W$, extend $v_{s+1}, \ldots, v_n$ to a basis $v_1, \ldots, v_s, v_{s+1}, \ldots, v_n$ of V. Then clearly every linear form which vanish on ${}^{\circ}W$ is a linear combination the vectors $v_1^*, \dots, v_s^*$ of the corresponding dual basis $v_1^*, \dots, v_n^*$ of V^*. In particular, $W \subseteq Kv_1^* + \dots + Kv_s^*$, and $\operatorname{Dim} W \leq s = \operatorname{Codim}({}^\circ W, V)$.

From 5.G.9 it follows immediately:

5.G.10 Theorem Let $W \subseteq V^*$ be a subspace of the dual space V^* of the finite dimensional K-vector space V. Then $({}^\circ W)^\circ = W$.

Proof Since $W \subseteq ({}^\circ W)^\circ$, it is enough to show that $({}^\circ W)^\circ \subseteq W$. But since ${}^\circ(({}^\circ W)^\circ) = {}^\circ W$, from 5.G.9 it follows that

$$\operatorname{Dim}({}^\circ W)^\circ = \operatorname{Codim}\left({}^\circ(({}^\circ W)^\circ), V\right) =$$

$$\operatorname{Codim}({}^\circ W, V) = \operatorname{Dim} W.$$

5.G.11 Example (Lagrange-Multipliers) Let $f_1, \dots, f_n$ be linear forms on the finite dimensional K-vector space V and let $W := Kf_1 + \cdots + Kf_n$ be subspace of V^* generated by $f_1, \ldots, f_n$ of the dual space V^* of V. Then

$${}^\circ W = (\mathrm{Ker} f_1) \cap \cdots \cap (\mathrm{Ker} f_n)$$
$$= \{ f_1 = f_2 = \cdots = f_n = 0 \} \subseteq V \text{ and}$$

$\mathrm{Codim}({}^\circ W, V) \leq n$. In particular, $f_1, \ldots, f_n$ are linearly independent if and only if $\mathrm{Codim}({}^\circ W, V) = n$. Therefore n linear forms, in general define, if they are linearly independent, an n-codimensional subspace.

By 5.G.10 every linear form f on V which vanish on ${}^\circ W$ is a linear combination $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$ of $f_1, \ldots, f_n$. Therefore, if $f_1, \ldots, f_n$ are linearly independent, then $\lambda_1, \ldots, \lambda_n \in K$ are uniquely determined and are called the Lagrange-Multipliers of f.

Further, from 5.G.10, we have:

5.G.12 Theorem Let V be a finite dimensional K-vector space and let W_1, W_2 be subspaces of the dual space V^* of V. Then $W_1 \subseteq W_2$ if and only if ${}^\circ W_1 \supseteq {}^\circ W_2$. In particular, $W_1 = W_2$ if and only if ${}^\circ W_1 = {}^\circ W_2$.

5.G.13 Remark The formula in 5.G.9 and hence also in 5.G.10 holds for a finite dimensional subspace W of the dual space V^* of an arbitrary K-vector space V, where we define
$\text{Codim}({}^\circ W, V) := \text{Dim}(V/{}^\circ W)$, for this see 6B, Exercise 1. It follows that in Example 5.G.11, the hypothesis that V is finite dimensional can be dropped and in 5.G.12 we need only to assume that W_1 and W_2 are finite dimensional.

5.G.14 Example (Bidual) Let V be a K-vector space. The space

$$V^{**} := (V^*)^*$$

is called the <u>bidual</u> of V.

There exists a cannonical homomorphism (evaluation map)

$$\sigma_V : V \longrightarrow V^{**}$$

defined by $x \longmapsto (e \longmapsto e(x))$.

Clearly $\operatorname{Ker} \sigma_V = {}^\circ(V^*) = 0$ and hence σ_V is injective. If V is <u>not</u> finite dimensional, then σ_V is <u>not</u> surjective, see Exercise 2c). But, if V is finite-dimensional, then $\operatorname{Dim} V^{**} = \operatorname{Dim} V^* = \operatorname{Dim} V$ by 5.G.4 and hence the injective homomorphism σ_V is bijective:

5.G.15 Theorem For every finite dimensional K-vector space, the cannonical homomorphism $\sigma_V : V \to V^{**}$ is an isomorphism.

With the help of σ_V, generally, one can identify a finite dimensional K-vector space V with its bidual V^{**}.

5.G.16 Example (Linear independence of functions) Let D be an arbitrary set and let $f_1, \dots, f_n \in K^D$ be K-valued functions on D. Let W denote the subspace of K^D generated by these functions $f_1, \dots, f_n$. The functions $f_1, \dots, f_n$ define the map $f : D \to K^n$ by $f(t) := (f_1(t), \dots, f_n(t))$, $t \in D$. With these notations, we have:

5.G.17 Theorem The following statements are equivalent:

(1) The functions $f_1, \dots, f_n$ are linearly independent in K^D.

(1') $\operatorname{Dim}_K W = n$

(2) The image $\operatorname{Im} f$ is a generating system of K^n.

(2') There exist elements $t_1, \dots, t_n \in D$ such that their images $f(t_i) = (f_1(t_i), \dots, f_n(t_i))$, $i = 1, \dots, n$, is a generating system (and hence basis) of K^n.

(3) There exists a subset $E \subseteq D$ of cardinality $|E| = n$ such that the restrictions $f_1|E, \ldots, f_n|E_n$ are linearly independent in K^E (and hence it follows that a basis of K^E)

(3') There exists elements $t_1, \ldots, t_n \in D$ such that the n-tuples $(f_j(t_1), \ldots, f_j(t_n))$, $j = 1, \ldots, n$, are linearly independent in K^n (and hence a basis of K^n).

(4) There exist functions $g_1, \ldots, g_n \in W$ and elements $t_1, \ldots, t_n \in D$ such that $g_j(t_i) = \delta_{ij}$ for $1 \le i, j \le n$.

<u>Proof</u> The equivalences $(1) \Longleftrightarrow (1')$, $(2) \Leftrightarrow (2')$ and $(3) \Leftrightarrow (3')$ are trivial, if one note that the elements $t_1, \ldots, t_n$ in (3') are pairwise distinct. Further, the implication $(3) \Rightarrow (1)$ is ~~trivial~~ trivial.

<u>$(1) \Longleftrightarrow (2)$</u>: Let U be the subspace generated by the Imf of K^n. By 5.G.7 $U = K^n$ if and only if the zero-linear form is the only linear form

on K^n which vanish on U. Therefore if $a_1 f_1(t) + \cdots + a_n f_n(t) = 0$ for all $t \in D$, then $a_1 = \cdots = a_n = 0$, i.e. $f_1, \ldots, f_n$ are linearly independent.

$\underline{(2') \Rightarrow (3)}$ The subset $E = \{t_1, \ldots, t_n\}$ of elements $t_1, \ldots, t_n$ given in (2') satisfies the condition in (3), since the image of $f|E$ is a generating system of K^n, the functions $f_1|E, \ldots, f_n|E \in K^E$ satisfies the condition in (2) and hence in (1) for $f_j|E$ instead of f_j.

$\underline{(3') \Rightarrow (4)}$ For the elements $a_{kj} \in K$, let $\sum_{k=1}^{n} a_{kj}\left(f_k(t_1), \ldots, f_k(t_n)\right) = e_j$, $j = 1, \ldots, n$, the standard basis of K^n. Then the functions $g_j = \sum_{k=1}^{n} a_{kj} f_k$, $j = 1, \ldots, n$, satisfy the condition in (4).

$\underline{(4) \Rightarrow (2)}$: The functions $g_1, \ldots, g_n \in W$ are linearly independent and hence $\operatorname{Dim} W \geq n$ and so $\operatorname{Dim} W = n$.

We proved the implications:

$$
\begin{array}{ccccccc}
(1) & \Rightarrow & (2) & & (3) & \Rightarrow & (1) \\
\Updownarrow & & \Updownarrow & \nearrow & \Updownarrow & & \\
(1') & & (2') & & (3') & \Rightarrow (4) \Rightarrow & (2).
\end{array}
$$

If $D=V$ is a K-vector space and the functions $f_1, \dots, f_n$ are linear, then $f: V \longrightarrow K^n$ is also linear and the image $\operatorname{Im} f$ is already a subspace of K^n. We note the following consequence of 5.G.17:

5.G.18 Corollary *Let V be a K-vector space. The linear forms $f_1, \dots, f_n \in V^*$ are linearly independent if and only if the homomorphism $f: V \longrightarrow K^n$ defined by $f(x) := (f_1(x), \dots, f_n(x))$ is surjective.*

In general, in the situation of 5.G.18, it follows that

$$\operatorname{Dim} \operatorname{Im} f = \operatorname{Dim}(Kf_1 + \cdots + Kf_n),$$

since $\operatorname{Ker} f = {}^\circ(Kf_1 + \cdots + Kf_n)$ by 5.G.9, see also Exercise 5.

Every linear map $f: V \longrightarrow W$ of K-vector spaces, in a canonical way, define a linear map

$$f^*: W^* \longrightarrow V^*, \quad e \longmapsto e \circ f$$

of the dual spaces. This map assigns each linear form $e \in W^*$, the composition $e \circ f \in V^*$

The map f^* is called the dual map corresponding to f. It is K-linear, which is a consequence of the computation rules in 5.C.2. Further,

$$(af+bg)^* = af^* + bg^*$$

for arbitrary $f, g \in \mathrm{Hom}_K(V, W)$ and all $a, b \in K$. Similarly,

$$(hf)^* = f^* \circ h^*$$

for K-linear maps $f: V \longrightarrow W$ and $h: W \longrightarrow X$. The most imortant is:

5.G.19 Theorem Let $f: V \longrightarrow W$ be a K-linear map of finite dimensional K-vector spaces and let $f^*: W^* \longrightarrow V^*$ be the corresponding dual map. Then

$$\mathrm{Rank}\, f = \mathrm{Rank}\, f^*.$$

Proof Let $v_1, \ldots, v_r \in V$ be elements such that $w_1 := f(v_1), \ldots, w_r := f(v_r)$ is a basis of $\mathrm{Im} f = f(V)$. Then $V = Kv_1 \oplus \cdots \oplus Kv_r \oplus \mathrm{Ker} f$. Let $v_i^* \in V^*$ be the linear form which vanish on $\mathrm{Ker} f$ and $v_i^*(v_j) = \delta_{ij}$, $1 \le i, j \le r$.

Then $v_1^*, \ldots, v_r^*$ are linearly independent. Further, for every $e \in W^*$, $f^*(e) = ef$ vanish on $\mathrm{Ker} f$ and $f^*(e)(v_i) = (ef)(v_i) = e(w_i)$ for $i = 1, \ldots, r$. Therefore

$$f^*(e) = \sum_{i=1}^{r} e(w_i) \cdot v_i^* \in \sum_{i=1}^{r} K v_i^*$$

Since $w_1, \ldots, w_r$ are linearly independent, for arbitrary $a_1, \ldots, a_r \in K$, there exists a linear form $e \in W^*$ such that $e(w_i) = a_i$, $i = 1, \ldots, r$. This proves that $\mathrm{Im} f^* = \sum_{i=1}^{r} K v_i^*$ and $\mathrm{Dim}_K \mathrm{Im} f^* =$

$r = \mathrm{Dim}_K \mathrm{Im} f$.

5.G.20 Remark The proof of 5.G.19 shows that: for arbitrary K-vector spaces V and W, we have: A linear map $f: V \longrightarrow W$ have finite rank if and only if the dual $f^*: W^* \longrightarrow V^*$ has finite rank. Moreover, in this case $\mathrm{Rank} f = \mathrm{Rank} f^*$ and $\mathrm{Im} f^*$ is the space of those linear forms on V which vanish on $\mathrm{Ker} f$ and hence $\mathrm{Im} f^* = (\mathrm{Ker} f)^0$. The equality $\mathrm{Ker} f^* = (\mathrm{Im} f)^0$ is trivial, see also Exercise 14.