6.B Quotient Spaces

We pass-on the construction of the residue-class groups given in the earlier subsection to K-vector space

Let K be a field, V be a K-vector space and $U \subseteq V$ be a K-subspace of V. Since $(U,+)$ is a subgroup of the additive group $(V,+)$ of V, the residue-class group

$$V/U := \{\bar{x} = x + U = U + x \mid x \in V\}$$

of the cosets

$$\bar{x} = x + U = U + x = \{u + x \mid u \in U\},$$

$x \in V$ (which are affine subspaces of V parallel to U). The group operation on V/U is precisely:

$$\bar{x} + \bar{y} := \overline{x+y} = (x+y) + U.$$

Moreover, the cosets $x + U$ are precisely the equivalence classes of the equivalence relation $\sim_U$ defined on V by: for $v, w \in V$, $v \sim_U w$ if $v - w \in U$.

Therefore V/U is the quotient-set $V/\sim_U$ of the equivalence relation $\sim_U$ and the quotient map $V \xrightarrow{\pi} V/U$ is a group homomorphism, i.e.

$\pi(x+y) = \pi(x) + \pi(y)$, where $\pi(x) = \overline{x}$.

Moreover, on the quotient-set V/U we shall define a scalar multiplication in a natural way:

For $a \in K$ and $x \in V$, we simply define $a \cdot \overline{x} := \overline{ax}$.

We again need to check that the product $a \cdot \overline{x}$ is independent of the choice of the representative x of the class $\overline{x}$. For this suppose that $\overline{x} = \overline{x'}$, i.e. $x - x' \in U$. Then also $ax - ax' = a(x-x') \in U$ and hence $\overline{ax} = \overline{ax'}$.

With this scalar multiplication of K on the abelian group V/U, it follows immediately by using the

Vector space axioms of V:

For $a, b \in K$ and $x, y \in V$, we have

(1) $a(b\overline{x}) = a(\overline{bx}) = \overline{a(bx)} = \overline{(ab)x}$

$= (ab)\overline{x}$

(2) $a(\overline{x}+\overline{y}) = a\overline{(x+y)} = \overline{a(x+y)} =$

$\overline{ax+by} = \overline{ax} + \overline{by} = a\overline{x} + b\overline{y}.$

(3) $(a+b)\overline{x} = \overline{(a+b)x} = \overline{ax+bx}$

$= \overline{ax} + \overline{bx} = a\overline{x} + b\overline{x}.$

(4) $1\cdot\overline{x} = \overline{1\cdot x} = \overline{x}.$

Therefore we have:

6.B.1 Theorem Let U be a subspace of the K-vector space V. Then the residue-class group V/U with the scalar multiplication

$$a(x+U) = (ax)+U,$$

$a \in K$, $x \in V$, is a K-vector space. The canonical projection $\pi : V \longrightarrow V/U$, $x \longmapsto \pi(x) := x+U$ is a surjective.

K-vector space homomorphism with the kernel $\operatorname{Ker}\pi = U$.
The K-vector space V/U is called the residue-class-space or the factor-space or the quotient-space of V modulo the subspace U.

The following theorem on induced homomorphisms and the isomorphism theorem are immediate:

6.B.2 Theorem Let $f: V \longrightarrow V'$ and $g: V \longrightarrow \overline{V}$ be homomorphisms of K-vector spaces with $\operatorname{Ker} f \supseteq \operatorname{Ker} g$. Further, assume that g is surjective. Then there exists a unique K-homomorphism $\overline{f}: \overline{V} \longrightarrow V'$ such that $\overline{f} \circ g = f$, i.e. the following diagramm

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g"'] & V' \\ \overline{V} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

is commutative.

Moreover, $\operatorname{Im} \overline{f} = \operatorname{Im} f$ and $\operatorname{Ker} \overline{f} = g(\operatorname{Ker} f)$.

6.B.3 Isomorphism-Theorem Let $f: V \longrightarrow V'$ be a homomorphism of K-vector spaces. Then f induces an isomorphism $\bar{f}: V/_{\mathrm{Ker} f} \longrightarrow \mathrm{Im} f$. In particular, K-vector spaces $V/\mathrm{Ker} f$, $\mathrm{Im} f$ are isomorphic, i.e. $V/_{\mathrm{Ker} f} \cong \mathrm{Im} f$.

From the Rank-Theorem the following dimension-formula is immediate:

6.B.4 Theorem Let V be a finite dimensional K-vector space and let U be a subspace of V. Then the residue-class space V/U is also finite dimensional and

$$\mathrm{Dim}_K V/U = \mathrm{Dim}_K V - \mathrm{Dim}_K U.$$

More generally, the residue classes $\overline{x_i}$, $i \in I$, of the elements $x_i \in V$, $i \in I$, form a basis of V/U if and only if the elements x_i, $i \in I$, form a basis of a complement of U in V. In particular, every complement of U in V is isomorphic to V/U.

For a subspace U of a vector space V, the dimension of V/U is called the <u>codimension of U in V</u> and is usually denoted by

$$\operatorname{Codim}(U,V) = \operatorname{Codim}_V U.$$

If $\operatorname{Codim}(U,V)$ is finite, then U is called <u>finite-codimensional</u> subspace of V. Therefore <u>U has the codimension $n \in \mathbb{N}$ in V if and only if U has n-dimensional complement in V</u>. The 1-codimensional subspaces of V are called the <u>hyperplanes in V</u>, see 5.G, Exercise 1. Since $V/\operatorname{Ker} f \cong \operatorname{Im} f$, it follows that $\operatorname{Rank} f = \operatorname{Codim}(\operatorname{Ker} f, V)$ for every K-linear map $f : V \longrightarrow W$. The residue-class space

$$\operatorname{Coker} f := W/\operatorname{Im} f$$

is called the <u>cokernel of f</u>. With this f is surjective if and only if $\operatorname{Coker} f = 0$.

6.B.5 Example (Residue-class algebras) Let A be a K-algebra and $\mathfrak{a} \subseteq A$ be an (two-sided) ideal in A, see Example 6.A.15. Then $\mathfrak{a}$ is also (since $ax = (a \cdot 1_A)x = x(a1_A)$ for $a \in K$) a K-subspace of A and the residue-class ring $A/\mathfrak{a}$ acquires a K-vector space structure. Since $(a\bar{x})(b\bar{y}) = (\overline{ax})(\overline{by}) = \overline{(ax)(by)} = \overline{(ab)(xy)} = (ab)(\overline{xy}) = (ab)(\bar{x}\bar{y})$ for $a, b \in K$, $x, y \in A$, $A/\mathfrak{a}$ is a K-algebra. It is called the residue-class algebra of A modulo the ideal $\mathfrak{a}$.

6.B.6 Remark (Residue-class modules) Let A be an arbitrary ring V be an A-module (see Remark 1.B.7) and $U \subseteq V$ be an A-submodule of V. Therefore $(U, +)$ is a subgroup of the additive group $(V, +)$ of V and it is closed under the scalar multiplication of A. Then the residue class group V/U acquires an A-module structure with the scalar multiplication $a \cdot \bar{x} := \overline{ax}$, $a \in A$, $x \in V$. This A-module V/U is called the residue-class module of V modulo the submodule U.

Exercises

In the following exercises, K denote a field and V denote a K-vector space.

1. Let $n \in \mathbb{N}$. A subspace $U \subseteq V$ of V has codimension n if and only if there exist n linearly independent linear forms $f_1, \ldots, f_n$ on V with $U = \bigcap_{i=1}^{n} \operatorname{Ker} f_i$.

2. Let $U_1, \ldots, U_n$ be finite-codimensional subspaces of V and $U := \bigcap_{i=1}^{n} U_i$. Further, let $U_i' := \bigcap_{j \neq i} U_j$, $i = 1, \ldots, n$.

a) U is finite-codimensional with

$$\operatorname{Codim}(U, V) \leq \sum_{i=1}^{n} \operatorname{Codim}(U_i, V)$$

b) The following statements are equivalent:

(i) The inequality in part a) is equality.

(ii) The canonical homomorphism

$$V/U \longrightarrow \bigoplus_{i=1}^{n} V/U_i$$ is an isomorphism.

(iii) $U_i + U_i' = V$ for $i = 1, \ldots, n$.

(iv) $U_1' + \cdots + U_n' = V$

(v) The sum $\sum_{i=1}^{n} U_i^\circ$ of subspaces U_i°, $i = 1, \dots, n$, in V^* is direct.

3. Let $f: V \longrightarrow W$ be a K-linear map of finite dimensional vector spaces. Then the equality

$\operatorname{Dim} \operatorname{Ker} f - \operatorname{Dim} \operatorname{Coker} f = \operatorname{Dim} V - \operatorname{Dim} W.$

4. Let U, W be subspaces of V with $U \subseteq W$. If W' is a complement of W in V, then $(U+W')/U$ is a complement of W/U in V/U, which is isomorphic to W'.

5. a) Let $f: V \longrightarrow V'$ be a K-linear map of vector spaces and $U \subseteq V$, $U' \subseteq V'$ be subspaces of V and V' respectively with $f(U) \subseteq U'$. Then f induces a K-linear map $\bar{f}: V/U \longrightarrow V'/U'$ with $\bar{f}(\bar{x}) = \overline{f(x)}$, $x \in V$.

b) Let U and U' be subspaces of the vector space V. Then the injective map $U \longrightarrow U+U'$ induces an isomorphism

(Noether's isomorphism):

$$U/_{U\cap U'} \cong U+U'/_{U'}.$$

c) Let $U, U' \subseteq V$ be subspaces of V with $U' \subseteq U$. Then the identity map $id_V : V \to V$ induces a K-vector space homomorphism $V/_{U'} \longrightarrow V/_{U}$ and this map induces an isomorphism $(V/_{U'})/_{(U/_{U'})} \cong V/_{U}$ of vector spaces.

6. Let $f : V \longrightarrow V$ be an operator on V. Then the following statements are equivalent

(i) f induces an automorphism of $\operatorname{Im} f$.

(ii) f induces an automorphism of $V/\operatorname{Ker} f$

(iii) $V = \operatorname{Im} f \oplus \operatorname{Ker} f$.

(iv) $\operatorname{Ker} f$ has an f-invariant complement $W \subseteq V$ such that $f|W : W \to W$ is an automorphism of W.

(For finite dimensional case, see 5.E, Exercise 7. -- The space W in (iv) is necessarily $\operatorname{Im} f$.).